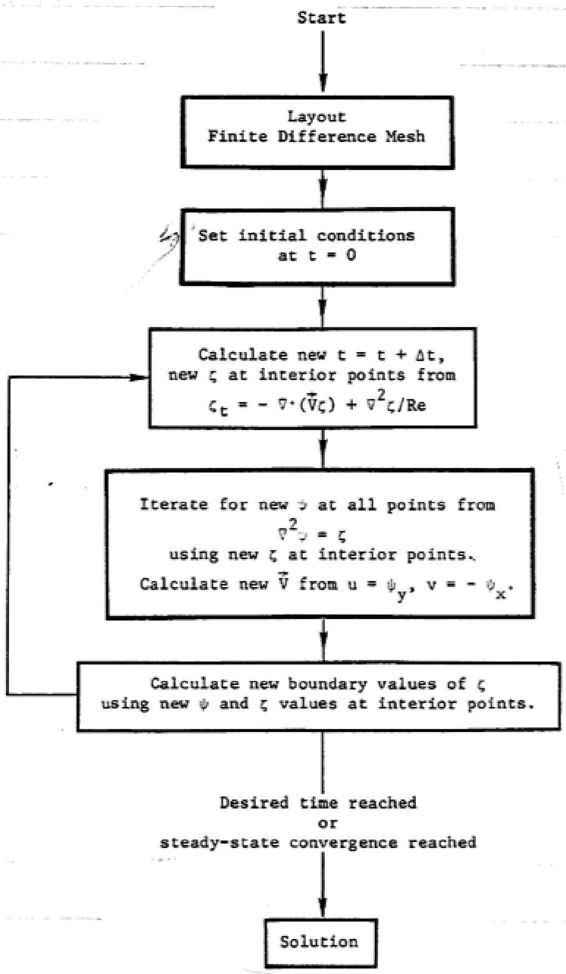
3. Basic Finite Difference Concepts

We concentrate at the finite-difference approach. Other methods will be stretched later. Now, first the framework in which we proceed to solve the equations of Chapter 2.

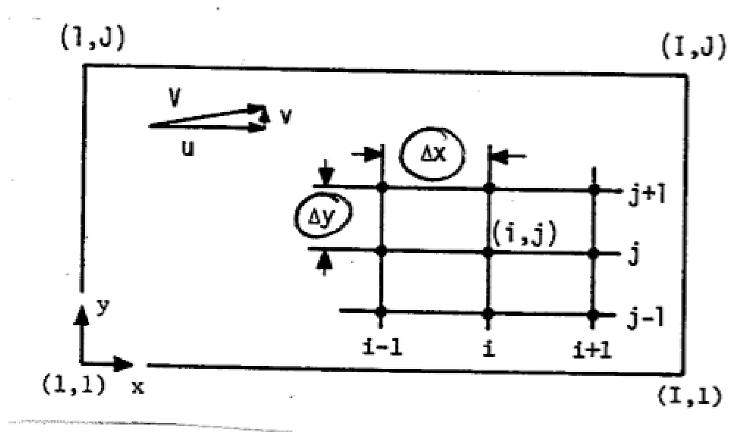
First a set of critical values ψ, ζ, μ, γ everywhere at time t=0. The computational cycle then starts with the use of a finite-difference equation for ζ to approximate $\frac{d\zeta}{dt}$. We then computer ζ at a new time level. Then we solve the Poisson equation for ψ which then gives us μ, γ and so on as depicted by this figure below.



3.1 Basic Finite - difference forms.

a. Taylor series expansions

- Rectangular Mesh



- Taylor series expansion is an interval about $x = a$.

(1) $f(x) = f(a) + f\prime(a)(x-a) + \dfrac{f\prime\prime(a)(x-a)^2}{2!} + \dfrac{f^{(n)}(a)(x-a)^{(n)}}{n!}$

Then the uncentered first derivative form of $\dfrac{\partial f}{\partial x}$ can then be expressed as a function of

$f_{i,j}$, $f_{i+1,j}$, $f_{i-1,j}$

Taylor series expansion $\longrightarrow$

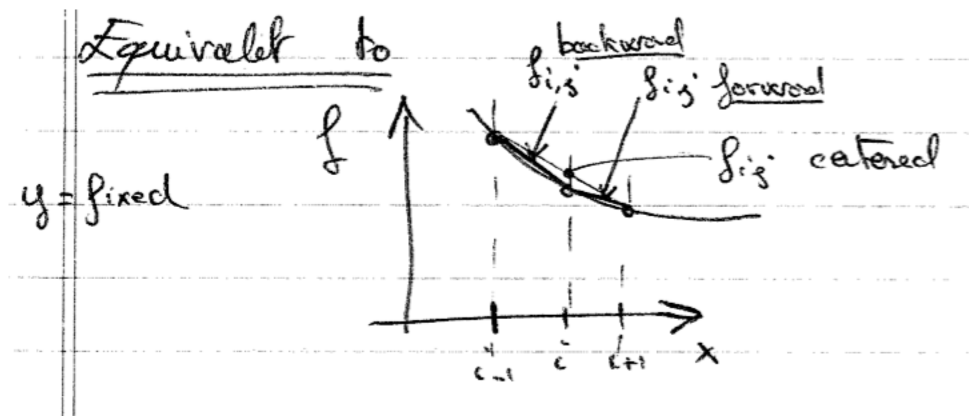
(2) $f_{CH,j} = f_{i,j} + \left.\dfrac{\partial f}{\partial x}\right|_{i,j} (x_{i+1,j} - x_{i,j}) + \dfrac{1}{2}\left.\dfrac{\partial^2 f}{\partial x^2}\right|_{i,j} (x_{i+1,j} - x_{i,j})^2 + \ldots$

or

$$f_{i+1,j} = f_{i,j} + \left.\frac{\partial f}{\partial x}\right|_{i,j} \Delta x + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{i,j} \delta x^2 + O(\Delta x^3)$$

$\longrightarrow \left.\dfrac{\partial f}{\partial x}\right|_{i,j} = \dfrac{f_{i+1,j} - f_{i,j}}{\Delta x} + \underline{O(\Delta x)} \leftarrow$ Terms of order Δx or first-order accuracy.

We can expand backwards which then gives
$(\frac{\partial f}{\partial x})_{i,j} = \frac{f_{i,j} - f_{i-1,j}}{\Delta x}$



The centered difference approximation $\frac{\partial f}{\partial x}$ is obtained by subtracting the forward and backwards expansions.

(6) $f_{i+1,j} = f_{i,j} + \left.\frac{\partial f}{\partial x}\right|_{i,j} \Delta x + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{i,j} \Delta x^2 + \frac{1}{6}\left.\frac{\partial^3 f}{\partial x^3}\right|_{i,j} \Delta x^3 + \frac{1}{24}\left.\frac{\partial^4 f}{\partial x^4}\right|_{i,j} \Delta x^4 + O(\Delta x^5)$

$f_{i-1,j} = f_{i,j} - \left.\frac{\partial f}{\partial x}\right|_{i,j} \Delta x + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{i,j} \Delta x^2 - \frac{1}{6}\left.\frac{\partial^3 f}{\partial x^3}\right|_{i,j} \Delta x^3 + \frac{1}{24}\left.\frac{\partial^4 f}{\partial x^4}\right|_{i,j} \Delta x^4 + O(\Delta x^5)$

$\longrightarrow f_{i+1,j} - f_{i-1,j} = 2\left.\frac{\partial f}{\partial x}\right|_{i,j} \Delta x + \frac{1}{3}\left.\frac{\partial^3 f}{\partial x^3}\right|_{i,j} \Delta x^3 + O(\Delta x^5)$
or
$\left.\frac{\partial f}{\partial x}\right|_{i,j,} = \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x} - \frac{1}{6}\left.\frac{\partial^3 f}{\partial x^3}\right|_{i,j} \Delta x^2 + O(\Delta x^4)$

(7) $\left.\frac{\partial f}{\partial x}\right|_{i,j} = \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x} + O(\Delta x^2) \longleftarrow$ Second-order accuracy

Analog expressions can be derived for y and t

(8) $\left.\frac{\partial f}{\partial y}\right|_{i,j} = \frac{f_{i,f+1} - f_{i,f-1}}{2\Delta y} + O(\Delta y^2)$

(9) $\left.\frac{\partial f}{\partial t}\right|_{i,j}^{n} = \frac{f_{ij}^{n+1} * - f_{ij}^{n-1}}{2\Delta t} + O(\Delta t^2)$

We can also derive an expression for $\dfrac{\partial^2 f}{\partial x^2}$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{i,j} = \frac{f_{i+1,j} + f_{i-1,j} - 2f_{ij}}{\Delta x^2} + O(\Delta x^2)$$

<u>Second Order Accurate</u>

<u>Polynomial fitting</u>

Another method of obtaining finite-difference expressions is to fit an analytical function with free parameters to mesh-pour* values and then to analytically differentiate the function.

Commonly, polynomials are used.

<u>Parabolic fit:</u> Data* at* $i, i+1, i-1$ for f

For convenience, $x = 0$ is at the location i

$f(x) = a + bx + cx^2$

$$\left| f_{i-1} = a - b\Delta x + c\Delta x^2 \right.$$

$$\left| f_i = a \right.$$

$$\left| f_{i+1} = a + b\Delta x + c\Delta x^2 \right.$$

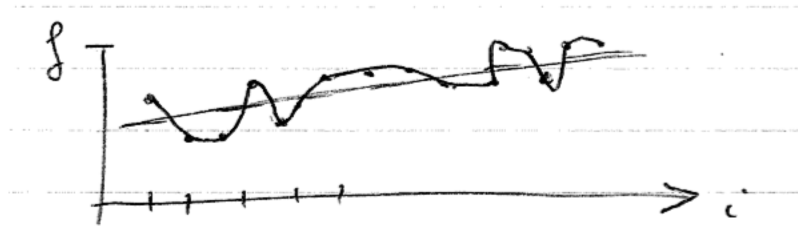
$\rightarrow c = \dfrac{f_{i+1} + f_{i-1} - 2f_i}{2\Delta x^2}$

$b = \dfrac{f_{i+1} - f_{i-1}}{2\Delta x}$

(11) $\rightarrow \left.\dfrac{\partial f}{\partial x}\right| = b$ and $\left.\dfrac{\partial^2 f}{\partial x^2}\right| = 2c$

which are obviously equivalent to the second order FD obtained in the previous section.

If we just use $y = ax + b$, then we obtained a first order *accuracy (forward and backward of the previous section). Higher polynomials give higher order. Beware of too high.



In general, a cubic spline* (polynomial) is often used since they indicate the presence of an uflexion* prout**.

c)<u>Integral Method</u>

In the integral method, we satisfy the governing equation in an integral *use, rather than a differential use*. We write the model equation in conservation* form

(12) $\frac{\partial \zeta}{\partial t} = -\frac{\partial(\mu\zeta)}{\partial x} + \alpha\frac{\partial^2\zeta}{\partial x^2}$

Integration from t to $t+\Delta t$ and $x-\frac{\Delta x}{2}$ to $x+\frac{\Delta x}{2}$

(13)$\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}})(\int_t^{t+\Delta t}\frac{\partial \zeta}{\partial t}dt)dx = -\int_t^{t+\Delta t}(\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}}\frac{\partial(\mu\zeta)}{\partial x}dx + k\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}}\frac{\partial^2\zeta}{\partial x^2}dx)dt \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}}(\zeta^{t+\Delta t} - \zeta^t)dx = -\int_t^{t+\Delta t}$

****FORMULA NOT READABLE FROM THIS POINT ON****

<u>Theorem: Mean Value Theorem</u>

$\left|\left|\int_{z1}^{z1+\Delta z} f(z)dz \exists f(\vec{z} * \Delta z \exists \vec{z} \in z_1, z_1+\Delta z\right.\right.$

Convergence is observed for $\Delta z \to 0$.

Using z at the lower integration limit (Euler's Integration) then (14) can be rewritten as

$\int_x^{t+\Delta t} - \int_x^{\Delta t}]\Delta x = -[(\mu\zeta)^t x + \frac{\Delta x}{2} - (\mu\zeta)^t x - \frac{\Delta}{2}\Delta t + \alpha[\frac{\partial\zeta}{\partial x}\Big|_{x+\frac{\Delta}{2}}^t - \frac{\partial\zeta}{\partial x}\Big|_{x-\frac{\Delta x}{2}}^t \Delta t$

The first derivatives can be evaluated as

$\zeta_{x+\Delta x}^t = \zeta_x^t + \int_x^{x+\Delta x}\frac{\partial\zeta}{\partial x}dx$

or

$$\left.\frac{\partial z}{\partial x}\right|_{x+\frac{\Delta x}{2}}^{t} = \frac{z^t_{x+\Delta x} - z^t_x}{\Delta x}$$

$$(\mu\zeta)^t_{x+\frac{\Delta x}{2}} = \frac{1}{2}[(\mu\zeta)^t_x + (\mu\zeta)^t_{x+\Delta x}] \rightarrow$$

$$(16)\ \frac{\zeta^{t+\Delta t}_x - \zeta^t_x}{\Delta t} = -\frac{(\mu\zeta)^t_{x+\Delta x} - (\mu\zeta)^t_{x-\Delta x}}{2\Delta x} + \alpha\frac{\zeta^t_{x+\Delta x} + \zeta_{x-\Delta x} - 2\zeta^t_x}{\Delta x^2}$$

Integration from $t - \Delta t$ to $t + \Delta t$ will give *catered in time

Advantage of this method is often appreciated in non-rectangular coordinate systems and because of the conservative property.

3.2 Trumcabion* errors, consistency, stability, and convergence

Suppose $\mu(x, t)$ is the exact solution to the initial value problem

$$(17)\ \frac{\partial \mu}{\partial t} = alpha(x, t)$$

and $U(n\Delta t, j\Delta x)$ is $= U^n_j$ is the solution to the FD approximation of (17). This approach must be $underlineconsistent$, $underlinestable$, and must $underlineconverge$ to be useful in physical problems.

Consistency: A FD approximation is consistent with a differential equation is the FD equation converges to the convect* differential equation as the space and time grid spacing* $\rightarrow 0$.

Stability: If U^n_j is the numerical solution and μ_j the exact solution at $t = n\Delta t$ and $x = j\Delta x$, then the FD approximation is stable of $Z^n_j = U^n_j - \mu^n_j$ remains founded as n trends to infinity for fixed Δt.

Convergence: If the difference between the theoretical solutions of FD and differential equations at a fixed point (x, t) trends to zero as $t \rightarrow 0$ and $\Delta x \rightarrow 0$ and $n, j \rightarrow \infty$ then the finite difference approximation converges to the continuous equation.

Trumbration* error: The local difference between the FD approximation and the Taylor series representation of the continuous problem as a fixed point is the *tr—- error.

Theorem: (Lax and Richtmyer)

Given a properly parcel linear initial value problem and a finite difference approximation to it that solidifies* the consistency condition, stability (as Δx and $\Delta t \rightarrow o$) is the necessary and sufficient condition for convergence.

Example:

Let's consider the one-dimensional advection* equation with constant speed c

$\frac{\partial \mu}{\partial t} + c\frac{\partial \mu}{\partial x} = 0.$

The Taylor series for second order derivatives are

$\mu_j^{n+1} - \mu_j^{n-1} = 2\Delta t(\frac{\partial \mu}{\partial t})\Big|_j^n + \frac{\Delta t^3}{3}(\frac{\partial^3 \mu}{\partial t^3})\Big|_j^n + ...$

$\mu_{f+}^n - \mu_{f-1}^n = 2\Delta x(\frac{\partial \mu}{\partial x})_f^n + \frac{\Delta x^3}{3}(\frac{\partial^3 \mu}{\partial x^3})\Big|_f^n + ...$

Combining we obtain

insert formula here

This FD approximation is consistent if the truscabian* error nehis** is $0(\Delta t^2 + \Delta x^2)$ goes to zero as $\Delta t, \Delta x \rightarrow 0$.

From (19)

$| E_j^n | \leq \frac{\Delta t^2}{126} M_1 + | c | \frac{\Delta x^2}{12} M_2$

where M_1 and M_2 are the bounds for $| \frac{\partial^3 \mu}{\partial t^3} |$ and $| \frac{\partial^3 \mu}{\partial x^3} |$ respectively. Note that these bounds hold for the true solution, i.e they are independent of the numerical treatment* of the equation. Therefore $E_f^n \rightarrow 0$ as $\Delta x, \Delta t \rightarrow 0$.

If we consider only finite-difference forward in Vines*, then

$| E_f^n | \leq \frac{\Delta t}{2} M_3 + | c | \frac{\Delta x^2}{12} M_4$ whose M_3 and M_4) are the bounds* for $| \frac{\partial^2 \mu}{\partial t^2} |$ and $| \frac{\partial^3 \mu}{\partial x^3} |$ respectively.

We are now interested in the accumulated error of FD solution. If we consider the latter (FD found in *line)

(20) $U_f^{n+1} = U_f^n - \frac{\lambda}{2}(U_{f+1}^n - U_{f-1}^n)$

(21) $\mu_f^{n+1} = \mu_f^n - \frac{\lambda}{2}(\mu_{f+1}^n - \mu_{f-1}^n) + \Delta t \xi_f^n$

with $\lambda = \frac{c\Delta t}{\Delta x}$

The <u>accumulated error</u> is $e_{f+1}^n - e_{f-1}^n) + \Delta t \varepsilon_j^n$

By substitution of (21) to (20),

(22) $e_f^{n+1} = e_f^n - \frac{\lambda}{2}(e_{f+1}^n - e_{f-1}^n) + \Delta t \varepsilon_j^n$

By defining $E^n = max_f \mid e_f^n \mid$ and $\varepsilon = max_{f,u} \mid \xi_f^n \mid$ then

$E^{n+1} \leq (1 + \mid \lambda \mid) E^n + \Delta t \varepsilon$

Successive use of this recursion* formula does NOT lead to a finite bound for E

$E^{n+1} \leq (1+ \mid \lambda \mid)[(1+ \mid \lambda \mid)E^{n-1} + \Delta t\varepsilon] + \Delta t\varepsilon$
$\leq ....$
$\leq [1 + (1+ \mid \lambda \mid) + (1+ \mid \lambda \mid)^i + ... + (1 + (\lambda 1)^{*u}]\Delta t\varepsilon$
if $E^0 = 0$
$\leq \frac{1+ \mid \lambda \mid)^n - 1}{\mid \lambda \mid}\Delta t\varepsilon$
$\leq \frac{\varepsilon \Delta x}{\mid c \mid}[(1 + \frac{\mid c \mid t}{n\Delta x})^n - 1]\frac{z\Delta x}{\mid c \mid}$ (with $\Delta t = \frac{t}{n}$
$\leq \frac{\varepsilon \Delta x}{\mid c \mid}(e\frac{\mid c \mid t}{\Delta x} - 1)$
which does to ∞ as $\Delta x \to 0$ and $n \to \infty$

Failure to find an upper limit for the error does not imply that this error will grow indefinitely. This can be done only by a practical test.

For this case, it turns out that an upper limit can be found if we replace U_f^n of (20) by $\frac{1}{2}(U_{f-1}^n + U_{f+1}^n)$

Then, instead of (22), we have
$e_f^{n+1} = (\frac{1}{2} + \frac{\lambda}{2}e_{f-1}^n + (\frac{1}{2} - \frac{\lambda}{2})e_{f+1}^n + \Delta t\varepsilon_f^n$
or
$E^n + 1 \leq (\mid \frac{1}{2} + \frac{\lambda}{2} \mid + \mid \frac{1}{2} - \frac{\lambda}{2} \mid)E^n + \Delta t\varepsilon$
As long as $\mid \lambda \mid \leq 1$ (CFL critoud*)
$E^{n+1} \leq E^n + \Delta t\varepsilon$
$\leq n\Delta t\varepsilon = t\varepsilon$

The accumulated error at a fixed time is then proportional to the trucation error *varepsilon*.

From Taylor series expansions
$\frac{i}{2}(\mu_{f-1}^n + \mu_{f+1}^n) = u_f^n + \frac{\Delta x^2}{4}((\frac{\partial^2 \mu}{\partial x^2})\Big|_f^n + (\frac{\partial^2 \mu}{\partial x^2})\Big|_j^n)$
The overall trucation* error can be bounded by
$\mid \varepsilon_f^n \mid \leq \Delta t\frac{M_1}{2} + \Delta x\frac{\mid c \mid M_2}{2\lambda} + \Delta x^2\frac{\mid c \mid M_3}{C}$

Where M_1, M_2, and M_3 are upper bounds for $\frac{\partial^2 \mu}{\partial t^2}\frac{\partial^2 \mu}{\partial x^2}, \frac{\partial^3 \mu}{\partial x^3}$ respectively.

Thus* the soleve is ** first $\lambda \neq 0$ and E, the accumulated error varesters* as the mesh width goes to zero.

$$\lim U_f^n = \mu(x,t)$$

$$\Delta x \to 0$$

$$\Delta t \to 0$$

$$\lambda < 0$$

This FD scheme* is then convergent

3.3 Norms and numerical stability analysis

a. Vector and matrix norms and stability definition

Stability is associated with the property of a numerical solution which remain finite at all points in the (x,t) domain. (Unstable $\leftrightarrow$ blow ups of the solution.)

A vector norm is defined as a measure of a vector in real-number space. The norm must satisfy

$$|| \vec{x} || \geq 0, \vec{x} || = 0, \leftrightarrow \vec{x} = \vec{0}$$
$$|| x\vec{x} || = | \alpha ||| \vec{x} || \text{ for any scalar } x$$
$$|| \vec{x} + \vec{y} || \leq || \vec{x} || + || \vec{y} || \text{ for any } \vec{x}, \vec{y}$$

A frequently used form is the Lp norm.

$$|| \vec{x} || = (\sum_{f=1}^{n} | xj |^p)^{\frac{1}{p}}$$

when $\vec{x} = (xj)$ is an n-dimensional vector. Most used* are:

(a) Euchidian norm, $p = 2$

L_∞(b) "mascinus" norm, p = ∞ $|| x ||_\infty = max_f | xj |$

(c) L_1) norm, p=1 $|| x ||_1 = \Sigma_f | xj |$

If we define $\vec{U}^n$ by $\vec{U}^n = (U_f^n)$, then a numerical scheme is stable if there exists a number M such that $|| \vec{U}^n || \leq M || \vec{U}^0 ||$ (M can be a function of time t since solutions grow in time)

By analogy with the definition of vector norms*, we define the matrix norm as a measure in real-number space. The following conditions must be satisfied:

$$|| A > 0, || A || = 0, \leftrightarrow A = (0)$$

$|| \alpha A ||=| \alpha ||| A ||$ for any scalar α
$|| A + B || \leq || A || + || B ||$
$|| A - B || \leq || A |||| B ||$ for any A, B.

The most cower* norms are as before the $L_1, L_2, and L_\infty$ norms.

$|| A ||_1 = Max_f \Sigma_i \mid aij \mid$ (Sum of all colors*)
$|| A ||_\infty = Max_i \Sigma_f \mid aij \mid$ (Sum of all norms*)
$|| A ||_2 = \sqrt{a(AtA)}$ where A is the absolute tangent eigurate* of the matrix AtA.

b) The Lax-Richtmyer Theorem

Theorem: Numerical* stability and consistency of a finite difference scheme imply convergence.

This theorem is important because it enables us to prove convergence of a numerical solution without explicit knowledge of the exact solution. The FD equation can be rewritten as
$\vec{U}^{n+1} = L\vec{U}^n + \vec{R}^n$
where L is a linear operator (expressed in a matrix form) and $\vec{R}^n$ is the in-homogenous part of the equation such as foraing*

Another definition of the stability, slightly more restrictive, let $f = r$ for most purposes, *especially in the following*. A finite FD scheme of the type of (26) is stable for any time t and any $S > 0$, there exists rwo* values ρ, n such that
$|| (L)^n || \leq M$ for all $\Delta x < \delta, \Delta t < y\Delta x$ and n provided that $n\Delta t \leq t$.

Since $\| \vec{U}^n \| \leq \| (L)^n \| \| \vec{U}^0 + \| (L)^{n-1} \| \| \vec{R}^0 \| + ... \| (L)^0 \| \| \vec{R}^{n-1} \|$

and since we can reasonably assume the total ferciy * $\Sigma_k \| R^k \|$ to be finite, this definition does imply *the are given before.

Proof of the LR Theorem

$\vec{U}^{n+1} = L\vec{U}^n + \vec{R}^n$
$\vec{\mu}^{n+1} = L\vec{\mu}^n + \vec{R}^n + \Delta t \vec{\varepsilon}^n$
The accumulated error vector $\vec{e}^{n+1}$ is then
$\vec{e}^{n+1} = Lvece^n + \Delta t \vec{\varepsilon}^n$
$= L(Le^{n-1} + \Delta t \vec{\varepsilon}^{n-1}) + \Delta t \vec{\varepsilon}^n$
$= ((L)^n \vec{\varepsilon}^0 + (L)^{n-1} \vec{\varepsilon}^1 + ...(\Delta^0 \vec{\varepsilon}^n)\Delta t$
$\rightarrow \| \vec{e}^n \| \leq \Delta t(\| (L)^{n-1} \| \| \vec{\varepsilon}^0 \| + ... + \| (L)^0 \| \| \varepsilon^{n-1} \|)$

since the scheme* is constant*, for every $\varepsilon > 0$, there exists two numbers δ, η such that $\parallel \vec{\varepsilon^k} \parallel < \varepsilon$ for all $\Delta x < \delta, \Delta t < \eta \Delta x$

Since the scheme is furthermore stable, we have $\parallel (L)^k \parallel \leq M$ for all $k, k\Delta t \leq t$,then

(27) $\parallel \vec{e^n} \parallel \leq n \Delta t \varepsilon M = t \varepsilon M$

Since ε is arbitrarily small, the theorem is proven.

The 2R theorem also holds in the opposite direction...convergence and consistency $\rightarrow$ stability.

C) Stability Analysis

The previous theorem allows us to concentrate on the stability of the numerical scheme *otter* then it's convergence, one you admit consistency. **

In section 3.2, we were not able to prove convergence of the scheme ***

$U_n^{f+1} = U_f^n - \frac{\lambda}{2}(U_{f+1}^n - U_{f-1}^n)$

1) <u>Using matrix norms</u>

The linear operator applicable in this case is

$$L = \begin{pmatrix} 1 & -\lambda/2 & & & \lambda/2 \\ \lambda/2 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -\lambda/2 \\ -\lambda/2 & & & \lambda/2 & 1 \end{pmatrix}$$

Amplification matrix

Since $\| L^n \| \leq \| L \|^n$ stability is assured if $\| L \| \leq 1$. Actually, $\| L \| \leq 1 + 0(\Delta t)$ is sufficient since

$\lim\limits_{n \to \infty} \| L \|^n \leq \lim(1 + \frac{0(f)}{n})^n = e^{0(t)})$

which is compatible with the previous definition. this criteria is named after Von Neumann.

We find that $\| L_1 \| = \| L_\infty \| = 1 + | \lambda |$.

Since $\lambda = \frac{c\Delta t}{\Delta x}$, the assumption $| \lambda | = 0(\Delta t)$ would imply $\Delta x = constant$. This is incompatible** with the limit process $\Delta x, \Delta t \rightarrow$ Hence matter** $L_1 or L_\infty$ can be used. The L_2 norm sequares* knowledge of the eiguvaules* of LTL.

The linear operator for the diffinive* scheme (24) is

$$L = \begin{pmatrix} 0 & (1/2 - \lambda/2) & & & (1/2 + \lambda/2) \\ (1/2 + \lambda/2) & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & (1/2 - \lambda/2) \\ (1/2 - \lambda/2) & & & (1/2 + \lambda/2) & 0 \end{pmatrix}$$

U_j^n is replaced by $\frac{1}{2}(U_{f-1}^n + U_{f+1}^n$

We find that

$\| L_3 \| = \| L \|_\infty = 1 y \mid \lambda \mid \leq 1, \mid \lambda \mid y \mid \lambda \mid > 1$******

Hence, in this case, stability is assured as long as $| \lambda | \leq 1$ (Same as convergence)

Let's now consider the following parabolic differential equation.

$$\frac{\partial F}{\partial t} = K\frac{\partial^2 F}{\partial x^2}$$

$$\frac{F_f^{u+1} - F_j^n}{\Delta t} = K\frac{F_{f+1}^n - 2F_f^n + F_{f-1}^n}{\Delta x^2}$$

or

$$F_f^{n+1} = \lambda F_{f-1}^n + (1-2\lambda)F_f^n + \lambda F_{f+1}^n, \text{ with } \lambda = \frac{K\Delta t}{\Delta x^2}$$

If the boundary values $F_o^n = F_J^n = 0$, then

$$\begin{pmatrix} F_1^{u+1} \\ \vdots \\ F_{1,g}^{u+1} \\ \vdots \\ F_{\sigma-1}^{u+1} \end{pmatrix} = \begin{pmatrix} 1-2\lambda & \lambda & & 0 \\ \lambda & 1-2\lambda & \lambda & \\ & \lambda & \ddots & \ddots \\ & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} F_1^u \\ \vdots \\ \vdots \\ F_{\sigma-1}^n \end{pmatrix}$$

(2) $F_n = LF_{n-1} = L^nF_o$, where L is an amplification matrix.

The eigeuvelues μ of L are the roots of

$| L - \mu I | -0$, where I is determined* of order J-1.

$\Rightarrow$ J-1 eiguenles**. Associated with each eigenwales is an eigu-vector v which satisfies $Lv_i = \mu_c v_i, c = 1, 2, ------$

Eigautras* $\Leftrightarrow$ base $\Rightarrow F_o = \Sigma_i C_i V_i$

$F_n = \Sigma_i C_i L^n v_i = \Sigma_i C_i L^{n-1} L v_i \leftarrow \mu_i V - i$

$= ... = \Sigma_i C_i \mu_i^n v_i$

Stable if $| \mu_i | \leq 1$ for all i.

Can be allowed for some growth novely

$| \mu_i | \leq 1 + o(\Delta t)$

(spectral radius)

Remember that our* scheme was not perfectly cascoteint* and $| \lambda |$ is bound away from 0. Both $| \lambda | < 1$ and $| \lambda | \geq \lambda_o > 0*$ must be simplified for convergence.

<u>Using Fourier* Methods</u> (or Van Newan* analysis)

The previous method is attractive, but often difficult to put into practice in more complicated situations. A less geuerd*, but simpler method is based on a Fourier* decomposition of solution U_f^n

$U_f^n = \sum_{k=-J}^{J} A^n L e^{ikxj}$

The exact solution is

(29)$\mu(x,t,) = \sum_{k=-n}^{n} Bk(t)e^{ikx}$

We can determine the amplibidies* $B_k(t)$ term by each $B_k(t)$ has then to satisfy

(30) $\frac{\partial B_k}{\partial t} = -ikcB_k$

or

(31) $B_k = a_k e^{-ikct}$ where $a_k = B_k(0)$ represents the initial conditions.

Let's now insert (28) in (22)

$U_f^{n+1} = \Sigma A_k^n e^{ikxj} - \frac{\lambda}{2}[\Sigma A_k^n(e^{ikxjh} - e^{ikxjh****})]$

(32) $= \Sigma A_k^n(1 - i\lambda \sin(k\Delta x)$

$= \Sigma Ak^{n+1} e^{ikxj}$

or $A_k^{n+1} = A_k^n(1 - i\lambda \sin(k\Delta X)$

The ratio $\frac{A_k^{n+1}}{A_k^n}$ is called the amplification* factor G.

(33) $G = 1 - i\lambda \sin(k\Delta x)$; $A_k^{n+1} = GA_k^n$

If solutions are to remain bound, then we have $| G | \leq 1$ (Van Newman*)

$| G |^2 = (1 - i\lambda \sin(k\Delta x))(1 + i\lambda \sin(k\Delta x)$

$= 1 + \lambda^2 \sin^2(k\Delta x)$

which shows that (22) is usable for all Δt

Exercise: Solve for deff** equation

- for both together.

Von Neunan* condition (more restricted)

$A_k^{n+1} = GA_k^n$

$= G^n A_k^o$ The scheme is stable y.

$| \mu_i | \leq 1 + O(\Delta t)$ for all i

where μ_i are the eigewalier*** of the amplification matrix G since we have

$(S_r^{(g)})^n \leq \| G^n \| \leq \| G \|^n$
(Richmyer*, See for details)