5. $\boxed{\text{Energetically Constant Finite Difference Schemes}}$

5.1 Nonlinear Stability

Inverted *nonlinear equation may become unstable even if the linear stability inversion* is not violated. After some time, "va***" (or noise) will appear on small scales, will grow slowly and eventually grow exponentially. In a linear problem, no Fourier models can interact with the other. One expect* the n*** to interact when nonlinearity is included $\rightarrow$ creation of variance. Since a uniform grid can only have wave numbers $k\epsilon[o, \frac{2\pi}{2\Delta x}]$ if any nonlinear interaction creates variance in scales $k > \frac{2\pi}{2\Delta x}$, the grid cannot resolve this energy and it will be folded into some other ucrewbles* (accumulation of small scales energy). The feed back through aliaxing* explains how nonlinear unstability can develope if energy is fadely* gathered* In numerical models, it is therefore important to damp out the small space scales to control nonlinear usabilities. This is done by getting rid of the accumication of energy in the small scales with an explicit viscosity (Leap—* Boharmonic*) or with a dissipation* finite difference method.

5.2 Energy Method

So far we have investigated the numerical stability of linear equations primarily by using the Fourier method. In the presence of non linear terms, to reduce the preuce* of non linear terms, the so called "energy method" is a powerful tool. This method may or may not have anything o do with physical forms of energy. It provides a sufficient condition for stability and is applicable to nonlinear equations.

If the true solution is known to be bounded, then the finite-diffrence solution should also be examined for boundaries. In other words, are quantities* that are conserved by the differential equations conserved by the FD equations?

a) Burger Equation

Lets first consider the simple case of the Burger equation ($\frac{\partial \mu}{\partial t} = \mu \frac{\partial \mu}{\partial x}$)

1) $\frac{\partial \mu}{\partial t} = -\mu \frac{\partial \mu}{\partial x}$

2) $\frac{\partial (\frac{\mu^2}{2}}{\partial t} = -\mu^2 \frac{\partial \mu}{\partial x} = -\frac{1}{3}\frac{2\mu^3}{\partial x}$

Integration with respect to x gives:

$$\int_o^L \frac{\partial \frac{\mu^2}{2}}{\partial t} dx = -\frac{1}{3}(\mu_L^3 - U_o^3)$$

which implies conservation of the kinetic energy if $U_o = U_L = 0$ (zero flux at the boundaries)

If the interval $[0, L]$ is discretized* in N regards* to Δx, then the RMS of (3) can be rewritten as :

$$-\frac{1}{3}[(U_1^2 - U_o^3) + (U_2^3 - U_1^3) + ...[U_L^3 - U_{n-1}]]$$

The integral is approximated by a sum with all neighbors canceling. Lets now examine several finite difference approximations.

*centered in space

$$\frac{\partial \mu j}{\partial t} = -\mu_f \frac{(\mu_{fH} - U_{f-1})}{2\Delta x}$$

Multiplying by μ_f and forming the sum (integral)

$\int_0^L \frac{\partial \frac{\mu^2}{2}}{\partial t} dx = -\frac{1}{2}\sum_{f=1}^{n-1}(\mu_f^2 \mu_{f+1} - U_j^2 U_{j-1}) \leftarrow$ This does not cancel and the scheme does not conserve energy.

Flex difference form

$$\frac{\partial \mu_j}{\partial t} = -\frac{\partial \frac{\mu_j^2}{2}}{\partial x} = -\frac{(\mu_{f+1}^2 - U_{f-1}^2)}{4\Delta x}$$

$\int_o^L \frac{\partial \frac{\mu^2}{2}}{\partial t} dx = -\frac{1}{4}\sum_{f=1}^{n-1}(\mu_{f+1}^2 U_f - U_{f-1}^2 \mu_f) \leftarrow$ does not cancel

Conserving scheme

$$\frac{\partial \mu_j}{\partial t} = -\frac{(\mu_{f+1} + U_f + U_{f-1})(\mu_{f+1} - U_{f-1})}{6\Delta x}$$

$$\int_0^L \frac{\partial \frac{\mu^2}{2}}{\partial t} dx = -\frac{1}{6}\sum_{f=1}^{n-1}(U_f^2 \mu_{f+1} + U_f U_{f-1}^2) - (U_{f-1}^2 U_f + U_{f-1} U_d^2)$$

which cancel.

This scheme is too simple for more complex systems.

b) Two-dimentional nonlinear advection* equation

$$\boxed{\frac{\partial \alpha}{\partial t} = -\vec{v} \cdot \nabla \alpha = -\mu \frac{\partial \alpha}{\partial x} - v \frac{\partial x}{\partial y}}$$

* x is a scalar which may depend on μ, v

(4) can be rewritten as:

$$(5) \quad \boxed{\frac{\partial \alpha}{\partial t} = -\nabla \cdot (\alpha \vec{v}) + \alpha \nabla \cdot \vec{v}}$$

For non divergent flow, the area average $\overline{\alpha}$ remains constant provided the spatial integration is done over a closed domain e. Under the same circumstances, any power of x is also conserved and in particular $\overline{\alpha^2}$

$$\frac{\partial}{\partial t}(\frac{\alpha^2}{2}) = \alpha(-\nabla \cdot (\alpha \vec{v}) + \alpha \nabla \cdot \vec{v})$$

$$= -\nabla \cdot (\alpha^2 \vec{v}) + \vec{v} \cdot \nabla \frac{\alpha^2}{2} + \alpha \nabla \cdot \vec{v}$$

$$= -\nabla \cdot (\alpha^2 \vec{v}) + \nabla \cdot (\frac{\alpha^2}{2}\vec{v}) - \frac{\alpha^2}{2}\nabla \cdot \vec{v} + \alpha^2 \nabla \cdot \vec{v}$$

$(6) = -\nabla \cdot (\frac{\alpha^2}{2}\vec{v}) + \frac{\alpha^2}{2}\underline{\nabla \cdot \vec{v}} \leftarrow= 0$

We then are trying to achive the same with the finite-difference expression of (5)

<u>Definition</u>:

$\delta_x a = \dfrac{a(x+\frac{\Delta x}{2}) - a(x - \frac{\Delta x}{2}}{\Delta x}$

$\overline{a}^x = \dfrac{a(z+\frac{\Delta x}{2} + a(x - \frac{\Delta x}{2})}{2}$

$\delta_x(\overline{a}^x) = (\overline{\delta_x a})^x$ commulative*

$\overline{a}^x \delta_x b = \delta_x(ab) - \overline{b}^x \delta_x a$
$a\delta_x b = \delta_x(\overline{a}^x b) - \overline{b\delta_x a}$

$\rightarrow \overline{a}^x \delta_x a = \delta_x \dfrac{a^2}{2}$

$\rightarrow a\delta_x \overline{a}^x = \delta x \dfrac{\overline{a^2}^x}{2}$

where $\overline{a^2}^x = a(x + \dfrac{\Delta x}{2} \cdot a(2 - \dfrac{\Delta x}{2})$

$\overline{ab}^x = \overline{a}^x + \overline{b}^x + (\dfrac{\Delta x}{2}\delta_x a)(\dfrac{\Delta x}{2}\delta_x b)$

$\overline{\overline{a}^x b}^x = q\overline{b}^x + \delta_x(\dfrac{\Delta x^2}{4} b\delta_x a)$

The following finite difference operator can be shown as "quadratic-conservative," i.w. that it leads to a FD conservation equation analog to (6) $\mu, v, \alpha)$ are defined at the same grid functions.

(7) $\boxed{\nabla \cdot (\alpha\vec{v}) = \delta_x(\overline{\alpha}^x\overline{\mu}^x) + \delta_y(\overline{\alpha}^y\overline{v}^y)}$