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5. Energetically Consistent Finite Difference Schemes

5.1 Nonlinear Stability

Inviscid nonlinear equation may become unstable even if the linear stability criterion is not violated. After some time "variance" (or noise) will appear on small scales, will grow slowly and eventually grow exponentially. In a linear problem, no Fourier modes can interact with the other, one expect the modes to interact when nonlinearity is included ⟹ creation of variance.

Since a uniform grid can only have wave numbers $k \in [0, \frac{2\pi}{2\Delta x}]$. If any nonlinear interaction creates variance in scales $k > \frac{2\pi}{2\Delta x}$, the grid cannot resolve this energy and it will be folded into some other wavenumbers. (Accumulation of small scales energy). The feed back through aliasing explains how nonlinear instability can develop if energy is falsely generated and persistently channeled toward the resolvable wavelengths. In numerical models, it is therefore important to damp out the small space scales to control nonlinear instabilities. This is done by getting rid of the accumulation of energy in the small scales with an explicit viscosity (Laplacian, Biharmonic, ...) or with a dissipative finite difference method.

5.2 Energy method

So far we have investigated the numerical stability of linear equations primarily by using the Fourier method. In the presence of non linear terms, to reduce the presence of nonlinear instabilities, the so-called "energy method" is a powerful tool.

this method may or may not have anything to do with physical forms of energy. It provides a sufficient condition for stability and is applicable to nonlinear equations.

If the true solution is known to be bounded, then ~~the~~ finite-difference solution should also be examined for boundedness. In other words are quantities that are conserved by the differential equations conserved by the FD equations?

a) Burgers equation

Lets first consider the simple case of the Burger equation $\left(\frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x}\right)$

$$(1) \quad \frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x}$$

$$(2) \quad \frac{\partial (u^2/2)}{\partial t} = -u^2 \frac{\partial u}{\partial x} = -\frac{1}{3} \frac{\partial u^3}{\partial x}$$

Integration with respect to x gives

$$(3) \quad \int_0^L \frac{\partial (u^2/2)}{\partial t} \, dx = -\frac{1}{3}\left(u_L^3 - u_0^3\right)$$ which

implies conservation of the kinetic energy if $u_0 = u_L = 0$ (zero flux at the boundaries)

If the interval $[0, L]$ is discretized in N segments Δx, then the RHS of (3) can be rewritten as

$$-\frac{1}{3}\left[(u_1^3 - u_0^3) + (u_2^3 - u_1^3) + \cdots \left[u_L^3 - u_{N-1}\right]\right]$$

The integral is approximated by a sum with all the neighbors canceling.

Lets now examine several finite difference approximations.

* centered in space

$$\frac{\partial u_j}{\partial t} = - u_j \frac{(u_{j+1} - u_{j-1})}{2 \Delta x}$$

Multiplying by u_j and forming the sum. (integral)

$$\int_0^L \frac{\partial (u^2/2)}{\partial t} dx = -\frac{1}{2} \underbrace{\sum_{j=1}^{N-1} \left(u_j^2 u_{j+1} - u_j^2 u_{j-1} \right)}$$

This does not cancel and
the scheme does not conserve energy

* flux difference form

$$\frac{\partial u_j}{\partial t} = - \frac{\partial (u_j^2/2)}{\partial x} = -\left(u_{j+1}^2 - u_{j-1}^2 \right) / 4 \Delta x$$

$$\int_0^L \frac{\partial (u^2/2)}{\partial t} dx = -\frac{1}{4} \sum_{j=1}^{N-1} \left(u_{j+1}^2 u_j - u_{j-1}^2 u_j \right)$$

does not cancel

* Conserving scheme

$$\frac{\partial u_j}{\partial t} = - \left(u_{j+1} + u_j + u_{j-1} \right) \left(u_{j+1} - u_{j-1} \right) / 6 \Delta x$$

$$\int_0^L \frac{\partial (u^2/2)}{\partial t} dx = -\frac{1}{6} \sum_{j=1}^{N-1} \left(u_j^2 u_{j+1} + u_j u_{j-1}^2 \right) - \left(u_{j-1}^2 u_j + u_{j-1} u_j^2 \right)$$

which cancel.

This scheme is too simple for more complex systems

b) Two-dimensional non linear advection equation

$$(4) \quad \boxed{\frac{\partial \alpha}{\partial t} = - \vec{V} \cdot \nabla \alpha = - u \frac{\partial \alpha}{\partial x} - v \frac{\partial \alpha}{\partial y}}$$

α is a scalar which may depend on u, v

(4) can be rewritten as

$$(5)\quad \boxed{\frac{\partial \alpha}{\partial t} = -\nabla\cdot(\alpha \vec{V}) + \alpha \underbrace{\nabla\cdot\vec{V}}_{=0}}$$

For non divergent flow, the area average $\overline{\alpha}$ remains constant provided the spatial integration is done over a closed domain. Under the same circumstances, any power of α is also conserved and in particular $\overline{\alpha^2}$

$$\frac{\partial}{\partial t}\left(\frac{\alpha^2}{2}\right) = \alpha\left(-\nabla\cdot(\alpha\vec{V}) + \alpha\,\nabla\cdot\vec{V}\right)$$

$$= -\nabla\cdot(\alpha^2\vec{V}) + \vec{V}\cdot\nabla\frac{\alpha^2}{2} + \alpha^2\,\nabla\cdot\vec{V}$$

$$= -\nabla\cdot(\alpha^2\vec{V}) + \nabla\cdot\left(\frac{\alpha^2}{2}\vec{V}\right) - \frac{\alpha^2}{2}\nabla\cdot\vec{V} + \alpha^2\,\nabla\cdot\vec{V}$$

$$(6)\quad = -\nabla\cdot\left(\frac{\alpha^2}{2}\vec{V}\right) + \frac{\alpha^2}{2}\underbrace{\nabla\cdot\vec{V}}_{=0}.$$

We then are trying to achieve the same with the finite-difference expression of (5)

Definitions: $$\delta_x a = \frac{a(x+\frac{\Delta x}{2}) - a(x-\frac{\Delta x}{2})}{\Delta x}$$

$$\overline{a}^x = \frac{a(x+\frac{\Delta x}{2}) + a(x-\frac{\Delta x}{2})}{2}$$

$$\delta_x(\overline{a}^x) = \overline{(\delta_x a)}^x$$ commutative

$$\overline{a}^x\,\delta_x b = \delta_x(ab) - \overline{b}^x\,\delta_x a$$
$$a\,\delta_x b = \delta_x(\overline{a}^x b) - \overline{b\,\delta_x a}^x$$
product rules

$$\Rightarrow \begin{cases} \overline{a}^x\,\delta_x a = \delta_x \dfrac{a^2}{2} \\[2ex] a\,\delta_x \overline{a}^x = \delta_x \dfrac{\widetilde{a^2}^x}{2} \end{cases}$$
where

$$\widetilde{a^2}^x = a(x+\tfrac{\Delta x}{2})\cdot a(x-\tfrac{\Delta x}{2})$$

$$\left\{\begin{array}{l} \overline{ab}^{x} = \bar{a}^{x}\,\bar{b}^{x} + \left(\frac{\Delta x}{2}\,\delta_x a\right)\left(\frac{\Delta x}{2}\,\delta_x b\right) \\ \overline{\bar{a}^{x} b}^{x} = a\,\bar{b}^{x} + \delta_x\left(\frac{\Delta x^2}{4}\, b\, \delta_x a\right) \end{array}\right.$$

The following finite difference operator can be shown as "quadratic - conservative", i, e that it leads to a FD conservation equation analog to (5) u, v, α are defined at the same grid locations.

$$(7) \quad \boxed{\nabla \cdot (\alpha \vec{v}) = \delta_x(\bar{\alpha}^{x}\,\bar{u}^{x}) + \delta_y(\bar{\alpha}^{y}\,\bar{v}^{y})}$$

Let's reproduce the steps that lead to (6)

$$\alpha\,\delta_x(\bar{\alpha}^{x}\bar{u}^{x}) = \delta_x\left[(\bar{\alpha}^{x})^2\,\bar{u}^{x}\right] - \overline{\bar{u}^{x}\,\bar{\alpha}^{x}\,\delta_x\alpha}^{x}$$

$$= \delta_x\left[(\bar{\alpha}^{x})^2\,\bar{u}^{x}\right] - \overline{\bar{u}^{x}\,\delta_x\frac{\alpha^2}{2}}^{x}$$

$$= \delta_x\left[(\bar{\alpha}^{x})^2\,\bar{u}^{x}\right] - \delta_x\left[\overline{\frac{\alpha^2}{2}}^{x}\,\bar{u}^{x}\right] + \frac{\alpha^2}{2}\,\delta_x\bar{u}^{x}$$

$$= \delta_x\left[\widetilde{\frac{\alpha^2}{2}}^{x}\,\bar{u}_x\right] + \frac{\alpha^2}{2}\,\delta_x\bar{u}^{x}$$

We have to add

$\alpha^2\,\nabla\cdot\vec{v} = \alpha^2\,\delta_x\,\bar{u}^{x} + \delta_y\,\bar{v}^{y}$ to arrive at (6)

$$\frac{\partial}{\partial t}\left(\frac{\alpha^2}{2}\right) = -\delta_x\left(\widetilde{\frac{\alpha^2}{2}}^{x}\,\bar{u}^{x}\right) - \delta_y\left(\widetilde{\frac{\alpha^2}{2}}^{y}\,\bar{v}^{y}\right) + \frac{\alpha^2}{2}\left(\delta_x\,\bar{u}^{x} + \delta_y\,\bar{v}^{y}\right)$$

which proves that $\frac{\alpha^2}{2}$ sum up to zero over all the grid points provided that the finite - difference divergence is equal to zero

$$(\delta_x\,\bar{u}^{x} + \delta_y\,\bar{v}^{y} = 0)$$

Let's now derive the FD equivalent to $\vec{v}\cdot\nabla\alpha$ which is consistent with the previous derivation.

$$\vec{v}\circ\nabla\alpha = \nabla\cdot(\alpha\vec{v}) - \alpha\,\nabla\cdot\vec{v}$$

$$u\frac{\partial\alpha}{\partial x} = \delta_x\left[\overline{\alpha}^x\,\overline{u}^x\right] - \alpha\,\delta_x\overline{u}^x$$

$$= \alpha\,\delta_x\overline{u}^x + \overline{\overline{u}^x\,\delta_x\alpha}^x - \alpha\,\delta_x\overline{u}^x$$

$$= \overline{\overline{u}^x\,\delta_x\alpha}^x$$

$$v\frac{\partial\alpha}{\partial y} = \overline{\overline{v}^y\,\delta_y\alpha}^y \qquad \text{Conservative}$$

Note that the "ordinary" advection operator

$$\vec{v}\circ\nabla\alpha = u\,\delta_x\overline{\alpha}^x + v\,\delta_y\overline{\alpha}^y$$

or

$$\nabla\cdot(\alpha\vec{v}) = \delta_x\left(\overline{\alpha u}^x\right) + \delta_y\left(\overline{\alpha v}^y\right)$$

do not arrive at an expression similar to (6). They are not quadratic-conservative, but are conservative, they conserve $\bar{\alpha}$ provided the divergence is equal to zero. (exercise)

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5.3 Energy and enstrophy conservation in the vorticity equation.

Non linear instability can develop is energy is falsely generated and persistently channelled toward the short resolvable wavelengths. Arakawa (1966) devised an elegant method for eliminating these artificial sources of energy. This can be illustrated by an important case of 5.2 by setting α equal to the vorticity

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

In two-dimensional flows, we can introduce the streamfunction ψ such that $\zeta = \nabla^2 \psi$ and $\vec{v} = \vec{k} \times \nabla \psi$

Equation (4) then becomes, as a function of ψ,

$$(8) \quad \boxed{\nabla^2 \frac{\partial \psi}{\partial t} = J(\nabla^2 \psi, \psi)}$$

where J is the Jacobian

Properties:

- $J(a,b) = a_x b_y - b_x a_y$

- $$J(a,b) = (\nabla a) \cdot \vec{k} \times \nabla b$$
$$= \nabla \cdot (\vec{k} \times a \nabla b)$$
$$= -\nabla \cdot (\vec{k} \times b \nabla a)$$

- Integration of the Jacobian over a domain D yields (Gauss law)

$$\iint_D J(a,b)\, dx\, dy = \oint a \frac{\partial b}{\partial s} ds = -\oint b \frac{\partial a}{\partial s} ds$$

$\Rightarrow$ the area integral vanishes if either a or b is constant along the edge of D. In particular, if (8) is integrated over a closed domain (the edge of which coincide with a streamline), then the area integral of $\nabla^2 \frac{\partial \psi}{\partial t}$ vanishes and $\overline{\zeta}$ is conserved

→ Multiplication of (8) by ζ yields.

(9) $$\frac{\partial}{\partial t}\left(\frac{\zeta^2}{2}\right) = J\left(\frac{\zeta^2}{2}, \psi\right)$$

which implies that $\overline{\frac{\zeta^2}{2}}$ is also conserved. $\frac{\zeta^2}{2}$ is called "enstrophy".

→ Multiplication of (8) by ψ yields:

(10) $$\psi \frac{\partial}{\partial t}(\nabla^2\psi) = J\left(\nabla^2\psi, \frac{\psi^2}{2}\right].$$

Integration over a closed domain implies that the RHS is equal to zero. The LHS term can be expressed as

$$\psi \frac{\partial}{\partial t}(\nabla^2\psi) = \nabla\cdot\left(\psi\nabla\frac{\partial\psi}{\partial t}\right) - \nabla\psi\cdot\left(\nabla\frac{\partial\psi}{\partial t}\right)$$

$$= \nabla\cdot\left(\psi\nabla\frac{\partial\psi}{\partial t}\right) - \frac{\partial}{\partial t}\frac{(\nabla\psi)^2}{2}$$

The area integral of $\nabla\cdot\left(\psi\nabla\frac{\partial\psi}{\partial t}\right)$ is then equal to $\underbrace{\psi_B}_{\text{on the boundary}} \oint \frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial n}\right) ds = \psi_B \oint \frac{\partial V_s}{\partial t} ds$ (11)

Addition of a constant to the ψ field does not affect the RHS of (10) or $\frac{\partial}{\partial t}\left(\frac{\nabla^{\cancel{2}}\psi^2}{2}\right)$ and therefore should not show any effect on the whole integral (11). Only true if $\oint \frac{\partial V_s}{\partial t} ds = 0$

Since $\frac{(\nabla\psi)^2}{2}$ is the kinetic energy, (8) then also conserve the mean kinetic energy in a closed domain. (Consistent with 5.2 since non divergent flow).

Let's now investigate if we can conserve enstrophy and energy in the FD operator representing the Jacobian $\sigma(\xi, \psi)$

$$\sigma(\xi,\psi) = \frac{\partial \xi}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \psi}{\partial x} \qquad (12)$$

$$= \frac{\partial}{\partial x}\left(\xi \frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\xi \frac{\partial \psi}{\partial x}\right) \qquad (13)$$

$$= \frac{\partial}{\partial y}\left(\psi \frac{\partial \xi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\psi \frac{\partial \xi}{\partial y}\right) \qquad (14)$$

We can then define

$$\sigma_1(\xi,\psi) = \left(\delta_x \overline{\xi}^{x}\right)\left(\delta_y \overline{\psi}^{y}\right) - \left(\delta_y \overline{\xi}^{y}\right)\left(\delta_x \overline{\psi}^{x}\right)$$

$$\sigma_2(\xi,\psi) = \delta_x\left(\overline{\xi\ \delta_y \overline{\psi}^{y}}^{x}\right) - \delta_y\left(\overline{\xi\ \delta_x \overline{\psi}^{x}}^{y}\right)$$

$$\sigma_3(\xi,\psi) = \delta_y\left(\overline{\psi\ \delta_x \overline{\xi}^{x}}^{y}\right) - \delta_x\left(\overline{\psi\ \delta_y \overline{\xi}^{y}}^{x}\right)$$

Since both σ_2 and σ_3 are of the form $\delta_x(\ ) + \delta_y(\ )$, they both conserve $\overline{\xi}$ (See 5.2). This is also true for σ_1

$$\sigma_1(\xi,\psi) = \delta_x\left[\left(\delta_y \overline{\psi}^{xy}\right)\overline{\xi}^{x}\right] - \overline{\overline{\xi}^{x}\delta_x\delta_y \overline{\psi}^{y}}^{x}$$

$$- \delta_y\left[\left(\delta_x \overline{\psi}^{xy}\right)\overline{\xi}^{y}\right] + \overline{\overline{\xi}^{y}\delta_x\delta_y \overline{\psi}^{x}}^{y}$$

$$= \delta_x(\qquad) - \delta_y(\qquad)$$

$$- \delta_x\left[\frac{\Delta x^2}{4}\left(\delta_x\delta_y \overline{\psi}^{y}\right)\delta_x \xi\right]$$

$$+ \delta_y\left(\frac{\Delta y^2}{4}\left(\delta_x\delta_y \overline{\psi}^{x}\right)\delta_y \xi\right) = \delta_x(\ ) + \delta_y(\ $$

Let's now investigate the quadratic conservative properties of σ_1, σ_2 and σ_3.

* It turns out that neither $\zeta\sigma_1$ nor $\psi\sigma_1$ can be brought into the form $\delta_x(\ ) + \delta_y(\ )$.

* $\boxed{\sigma_2}$

$$\zeta\,\sigma_2(\zeta,\psi) = \delta_x\left(\overline{\zeta}^x\,\overline{\zeta\delta_y\overline{\psi}^y}^x\right) - \overline{\overline{\zeta\delta_y\overline{\psi}^y}^x\,\delta_x\zeta}^x$$

$$- \delta_y\left[\overline{\zeta}^y\,\overline{\zeta\,\delta_x\overline{\psi}^x}^y\right) + \overline{\overline{\zeta\delta_x\overline{\psi}^x}^y\,\delta_y\zeta}^y$$

$$= \delta_x\left(\overline{\zeta}^x \cdots\right) - \underbrace{\zeta(\delta_y\overline{\psi}^y)(\delta_x\overline{\zeta}^x)} - \delta_x(\cdots$$

$$- \delta_y\left(\overline{\zeta}^y \cdots\right) + \underbrace{\zeta(\delta_x\overline{\psi}^x)(\delta_y\overline{\zeta}^y)}_{=-\zeta\sigma_1} + \delta_y(\cdots)$$

$$\Rightarrow \quad \zeta(\sigma_1+\sigma_2) = \delta_x(\cdots) + \delta_y(\cdots).$$

or $\quad \zeta\left(\frac{\sigma_1+\sigma_2}{2}\right) = \quad$ " $\quad$ "

$\Rightarrow$ $\boxed{\text{The average of } \sigma_1 \text{ and } \sigma_2 \text{ conserves enstrophy.}}$

$$\psi\,\sigma_2(\zeta,\psi) = \delta_x\left[\overline{\psi}^x\,\overline{\zeta\delta_y\overline{\psi}^y}^x\right] - \overline{\overline{\zeta\delta_y\overline{\psi}^y}^x\,\delta_x\psi}^x$$

$$- \delta_y\left(\overline{\psi}^y\,\overline{\zeta\delta_x\overline{\psi}^x}^y\right] - \overline{\overline{\zeta\delta_x\overline{\psi}^x}^y\,\delta_y\psi}^y$$

$$= \delta_x(\cdots) - \underbrace{\zeta(\delta_y\overline{\psi}^y)(\delta_x\overline{\psi}^x)}\ \delta_x(\ )$$

$$- \delta_y(\quad) + \underbrace{\quad''\quad}_{=0} + \delta_y(\cdots)$$

$$= \delta_x(\cdots) + \delta_y(\cdots)$$

$\boxed{\sigma_2 \text{ conserves energy}}$

$\boxed{J_3}$

Since σ_2 and σ_3 are antisymmetric with respect to ζ and ψ, it can be immediately concluded that

$$\boxed{\begin{array}{lll} \dfrac{\sigma_1+\sigma_3}{2} & \text{conserves} & \text{energy} \\ \sigma_3 & \text{conserves} & \text{enstrophy.} \end{array}}$$

and (the second line above)

We have now two energy conserving Jacobians

$$J_2 \,, \quad \frac{\sigma_1+\sigma_3}{2}$$

and two enstrophy conserving ones

$$\sigma_3 \,, \quad \frac{\sigma_1+J_2}{2}$$

Any linear combination of the two are also conserving (energy for σ_2, $\frac{\sigma_1+\sigma_3}{2}$
enstrophy for σ_3, $\frac{\sigma_1+\sigma_2}{2}$)

<u>Question:</u>

Is it then possible to find a Jacobian which is at the <u>same</u> time, a combination of J_2, $\frac{\sigma_1+\sigma_3}{2}$ <u>and</u> J_3 and $\frac{\sigma_1+\sigma_2}{2}$?

$$\alpha J_2 + \beta \frac{\sigma_1+J_3}{2} = \gamma J_3 + \delta \frac{\sigma_1+\sigma_2}{2}$$

with $\begin{pmatrix} \alpha+\beta=1 \\ \gamma+\delta=1 \end{pmatrix}$

There is one non-trivial solution

$$\alpha = \gamma = 1/3$$
$$\delta = \beta = 2/3$$

$$\Rightarrow \boxed{J_A = 1/3 \, (\sigma_1 + \sigma_2 + \sigma_3)} \qquad (15)$$

which is the Arakawa Jacobian.
which conserves <u>both</u> energy and enstrophy

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- For practical use, the Jacobians can be simplified.

$$(J_1+J_2)(\zeta,\psi) = \left((\delta_x \overline{\zeta}^x)\,\delta_y \overline{\psi}^y + \delta_x \left(\overline{\zeta\, \delta_y \overline{\psi}^y}\right)^x \right)$$

$$- \left((\delta_y \overline{\zeta}^y)\,\delta_x \overline{\psi}^x + \delta_y \left(\overline{\zeta\, \delta_x \overline{\psi}^x}\right)^y \right)$$

$$= \left(\delta_x \left(\overline{\zeta}^x\, \delta_y \overline{\psi}^{xy} \right) - \overline{\overline{\zeta}^x \delta_x \delta_y \overline{\psi}^y}^x + \overline{\overline{\zeta}^x \delta_x \delta_y \overline{\psi}^y}^x + \overline{(\delta_x \zeta)\delta_y \overline{\psi}^{xy}}^x \right)$$

$$- \left(\quad - - - - - - - \quad \right)$$

$$= 2\,\delta_x \left(\overline{\zeta}^x\, \delta_y \overline{\psi}^{xy} \right) - \zeta\, \delta_x \delta_y \overline{\psi}^{xy}$$

$$- 2\,\delta_y \left(\overline{\zeta}^y\, \delta_x \overline{\psi}^{xy} \right) + \zeta\, \delta_x \delta_y \overline{\psi}^{xy}$$

Enstrophy conserving

$$\Rightarrow \boxed{\frac{\sigma_1+\sigma_2}{2}(\zeta,\psi) = \delta_x\left(\overline{\zeta}^x \delta_y \overline{\psi}^{xy}\right) - \delta_y\left(\overline{\zeta}^y \delta_x \overline{\psi}^{xy}\right)} \quad (16)$$

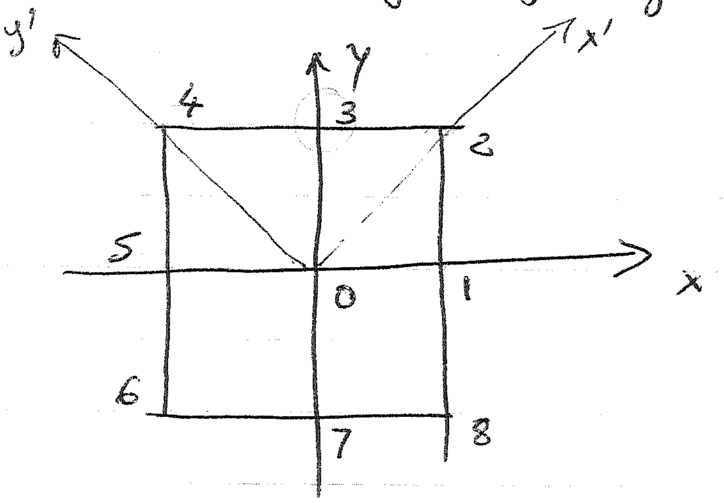
In a similar way;

Energy conserving

$$\boxed{\frac{J_1+J_3}{2}(\zeta,\psi) = -\delta_x\left(\overline{\psi}^x \delta_y \overline{\zeta}^{xy}\right) + \delta_y\left(\overline{\psi}^y \delta_x \overline{\zeta}^{xy}\right)} \quad (17)$$

- Expression for the Arakawa Jacobian:

Let's consider the following grid



J_3 can be written in the form

(18) $$4\Delta_x\Delta_y\, J_3(\zeta,\psi) = \Big(-\psi_1(\zeta_2-\zeta_8) + \psi_5(\zeta_4-\zeta_6)$$
$$+\ \psi_3(\zeta_2-\zeta_4) - \psi_7(\zeta_8-\zeta_6)\Big)$$

by subtracting ζ_0

$$=\Big((\zeta_2+\zeta_0)(\psi_3-\psi_1) - (\zeta_6+\zeta_0)(\psi_5-\psi_7)\Big)$$
$$-\Big((\zeta_4+\zeta_0)(\psi_3-\psi_5) - (\zeta_8+\zeta_0)(\psi_1-\psi_7)\Big)$$

(19) $$\Longrightarrow \boxed{J_3(\zeta,\psi) = \delta_{x'}(\overline{\zeta}^{x'}\delta_y\psi) - \delta_{y'}(\overline{\zeta}^{y'}\delta_{x'}\psi)}$$

coordinates axes (x',y') rotated 45° relative to (x,y)

$\Longrightarrow$ Combining (16) and (19)

$$\boxed{\begin{aligned} J_A(\zeta,\psi) = \frac{1}{12\Delta_x\Delta_y}\Big(\, & (\zeta_1+\zeta_0)(\psi_2+\psi_3-\psi_7-\psi_8) \\ & + (\zeta_3+\zeta_0)(\psi_4+\psi_5-\psi_1-\psi_2) \\ & + (\zeta_5+\zeta_0)(\psi_6+\psi_7-\psi_3-\psi_4) \\ & + (\zeta_7+\zeta_0)(\psi_8+\psi_1-\psi_5-\psi_6) \\ & + (\zeta_2+\zeta_0)(\psi_3-\psi_1) \\ & + (\zeta_4+\zeta_0)(\psi_5-\psi_3) \\ & + (\zeta_6+\zeta_0)(\psi_7-\psi_5) \\ & + (\zeta_8+\zeta_0)(\psi_1-\psi_7)\ \Big)\end{aligned}}$$

Arakawa Jacobian

The terms in ζ_0 cancel, the expression can use ζ_0, $-\zeta_0$ or no ζ_0.

The fact that one conserve both energy <u>and</u> enstrophy <u>eliminate</u> the aliasing problem caused by inhibited pile-up of energy near the short-wave resolution limit of the computational mesh since the <u>average</u> wave number is conserved.
In this particular case, one does not need to have an artificial diffusing scheme or a selective dissipative finite difference scheme.

Conservation of both mean enstrophy and energy greatly inhibits the redistribution of energy in wave number space. (Phillips, 1956, 1959; Fjørtoft, 1953 Arakawa, 1966)

Demonstration:

We decompose the ψ field into a series of orthogonal functions ψ_k $\quad \psi = \sum_k \psi_k$

The ψ_k ~~can~~ be chosen as the solutions of the Helmholtz equation $\boxed{\nabla^2 \psi_k + \lambda_k^2 \psi_k = 0}$ which form a complete set of orthogonal functions.

Let's consider the special case of a rectangular domain $\left\{ \begin{array}{l} 0 \leq x \leq a \\ 0 \leq y \leq b \end{array} \right.$

<u>Assumption</u>: separation of variables $\psi = A(x)\, B(y)$

$$\Rightarrow \quad A''B + B''A + \lambda^2 AB = 0$$

or

$$\frac{A''}{A} + \frac{\lambda^2}{2} = -\left(\frac{B''}{B} + \frac{\lambda^2}{2}\right) = \text{constant} = \gamma^2$$

Thus A and B must satisfy

$$A'' = -\left(\frac{\lambda^2}{2} - \gamma^2\right) A \quad \text{and} \quad B'' = -\left(\frac{\lambda^2}{2} + \gamma^2\right) B$$

If $A(0) = A(a) = B(0) = B(b) = 0$
(Homogeneous boundary conditions)

Then λ and γ must satisfy

$$\left|\begin{array}{l} \dfrac{\lambda^2}{2} - \gamma^2 = \dfrac{\pi^2 m^2}{a^2} \\[2ex] \dfrac{\lambda^2}{2} + \gamma^2 = \dfrac{\pi n^2}{b^2} \end{array}\right.$$

or $\boxed{\lambda_{n,m}^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)}$ with the corresponding eigenfunctions

$$\boxed{\psi_{n,m}(x,y) = \sin\pi\frac{mx}{a}\,\sin\pi\frac{ny}{b}}$$

→ The mean kinetic energy $\overline{K} = 1/2\,\overline{(\nabla\psi)^2}$ can be expressed as

$$\overline{K} = 1/2\,\overline{(\nabla\psi)^2} = 1/2\,\overset{=0\ \text{(average over closed domain)}}{\cancel{\overline{\nabla\cdot(\psi\nabla\psi)}}} - 1/2\,\overline{\psi\nabla^2\psi}$$

$$= 1/2\,\overline{\sum\psi_{m',n'}\sum\lambda^2_{mn}\psi_{m,n}}$$

$$= 1/2\sum\lambda^2_{m,n}\,\overline{\psi^2_{m,n}} = \sum_{n,m}K_{m,n}$$

where we used <u>the orthogonality</u> property of $\psi_{m',n'}\psi_{m,n} = 0$ unless $\left(\begin{array}{l} m'=m \\ n'=n\end{array}\right.$

→ The mean enstrophy ~~can~~ $\overline{\xi}^2/2 = \overline{E}$ can be expressed as.

$$1/2\,\overline{\xi}^2 = 1/2\,\overline{(\nabla^2\psi)^2} = 1/2\,\overline{\sum\lambda^2_{m,n}\psi_{mn}\sum\lambda^2_{m',n'}\psi_{m'n'}}$$

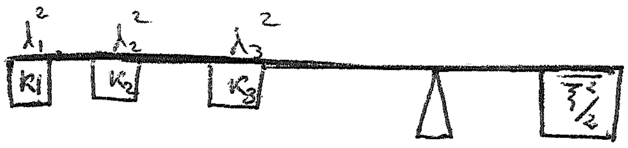
$$= 1/2\sum\lambda^4_{m,n}\,\overline{\psi^2_{m,n}} = \sum_{n,m}\lambda^2_{m,n}K_{m,n}$$

Therefore, we can derive an average wave number by

$$\boxed{\overline{\lambda^2} = \frac{\sum\lambda^2_{mn}K_{m,n}}{\sum K_{m,n}} = \frac{1/2\,\overline{\xi}^2}{\overline{K}}}$$

This wavenumber is conserved as long as $\overline{K}$ and $\frac{1}{2}\overline{\zeta}^2$ are conserved. The implication is that there can be no overall one-way cascade of energy in wave number space. If some "local" cascading takes place in some part of the spectrum, there must be a compensating shift of energy in another part of the spectrum.

This mechanism is often explained in terms of kinetic energies balancing a fixed amount of total enstrophy



If energy is transfered for example from K_2 to K_3 then the balance is broken. It will be balanced only if some energy is transfered also to K_1

Aliasing problems caused by pile-up of energy near the short wave resolution limit in the computational mesh can therefore be eliminated by using an <u>energy</u> and <u>enstrophy</u> conserving operator.

5.4. Quadratic conservative momentum advection schemes

The Jacobian operator $J(\zeta, \psi)$ of (8) arised through the cross-differentiation of the advection terms in the u and v momentum equations

$$\frac{\partial}{\partial x}\left(\vec{v}\cdot\nabla u\right) - \frac{\partial}{\partial y}\left(\vec{v}\cdot\nabla v\right)$$

It would be of great interest to backtrack this derivation to obtain finite difference operators for $\vec{v}\cdot\nabla u$ and $\vec{v}\cdot\nabla v$ which have conservative properties similar to the previous Jacobian operators.

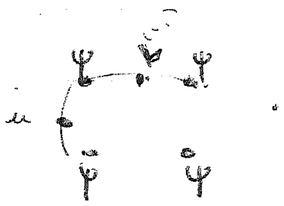
$$\frac{\partial}{\partial x}(\vec{v}\cdot\nabla v)-\frac{\partial}{\partial y}(\vec{v}\cdot\nabla u)=$$

$$\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)-\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)$$

(20) $$=\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}-u\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(v\frac{\partial v}{\partial x}-v\frac{\partial u}{\partial y}\right)$$

$$=\nabla\cdot[\vec{v}\,\zeta]=\vec{v}\cdot\nabla\zeta+\zeta\cdot\nabla v$$

We now define $\begin{cases}u=-\delta_y\psi\\ v=\delta_x\psi\end{cases}$



(Staggered u, v)

We can then rewrite σ_2 and $\frac{\sigma_1+\sigma_2}{2}$

$\ast$ $$\sigma_2(\zeta,\psi)=\delta_x\left(\overline{\zeta\;\delta_y\bar{\psi}^y}^x\right)-\delta_y\left(\overline{\zeta\;\delta_x\bar{\psi}^x}^y\right)$$

$$=-\delta_x\left(\overline{(\delta_x v-\delta_y u)\,\bar{u}^y}^x\right)-\delta_y\left(\overline{(\delta_x v-\delta_y u)\,\bar{v}^x}^y\right)$$

$\ast$ $$\frac{\sigma_1+\sigma_2}{2}(\zeta,\psi)=-\delta_x\left(\overline{(\delta_x v-\delta_y u)}^x\,\bar{u}^{xy}\right)-\delta_y\left(\left(\overline{\delta_x v-\delta_y u}\right)^y\,\bar{v}^{xy}\right)$$

$\ast$ $J_2(\zeta,\psi)$ can be rewritten as

(21) $$-\delta_x\left(\overline{\bar{u}^y\delta_x v}^x+\overline{\bar{v}^y\delta_y v}^y\right)+\delta_y\left[\overline{\bar{u}^x\delta_x u}^x+\overline{\bar{v}^x\delta_y u}^y\right]$$

since $\delta_x(\bar{u}^y\delta_y u)=\delta_y(\bar{v}^x\delta_x u)$

Comparing (21) to (20), we can deduce the following <u>energy</u>-conserving momentum advection operators.

(22)
$$\begin{cases} \vec{V}\cdot\nabla u = \overline{\bar{u}^{x}\delta_x u}^{x} + \overline{\bar{v}^{x}\delta_y u}^{y} \\ \vec{V}\cdot\nabla v = \overline{\bar{u}^{y}\delta_x v}^{x} + \overline{\bar{v}^{y}\delta_y v}^{y} \end{cases}$$

(Equivalent to the "semi-momentum" operators introduced in 5.2, except staggered in space)

In a similar manner, $\frac{J_1+J_2}{2}(\zeta,\psi)$ can be rewritten as:

(23)
$$-\delta_x\left(\bar{u}^{xy}\delta_x\bar{v}^{x} + \bar{v}^{yy}\delta_y\bar{v}^{y}\right] + \delta_y\left(\bar{u}^{xx}\delta_x\bar{u}^{x} + \bar{v}^{xy}\delta_y\bar{u}^{y}\right)$$

since $\delta_x\left(\bar{u}^{xy}\ \delta_y\bar{u}^{x}\right) - \delta_y\left(\bar{u}^{xx}\ \delta_x\bar{u}^{x}\right)$

We can then deduce the following enstrophy-conserving momentum advection operators.

(24)
$$\begin{cases} \vec{V}\cdot\nabla u = \bar{u}^{xx}\delta_x\bar{u}^{x} + \bar{v}^{xy}\delta_y\bar{u}^{y} \\ \vec{V}\cdot\nabla v = \bar{u}^{xy}\delta_x\bar{v}^{x} + \bar{v}^{yy}\delta_y\bar{v}^{y} \end{cases}$$

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→ If either (22)(24) is used in the momentum equation and if the flow is non divergent ($\delta_x u+\delta_y v=0$) then energy or enstrophy is conserved in the same manner as it is in the vorticity equation through J_2 or $\frac{J_1+J_2}{2}$. Stated differently, only the divergent part of the the velocity field is capable of creating or destroying energy or enstrophy, in perfect analogy to the behavior of ~~the~~ solutions of the differential equations.

We would like to have an operator that conserve both energy and enstrophy as the Arakawa Jacobian. This means converting J_3 and is slightly more complex.

We start from (19) which is the expression with the axes rotated 45°.

$$J_3(\zeta,\psi) = \delta_{x'}(\overline{\zeta}^{x'}\delta_{y'}\psi) - \delta_{y'}(\overline{\zeta}^{y'}\delta_{x'}\psi)$$

If we define $u' = -\delta_{y'}\psi$ and $v' = \delta_{x'}\psi$, then J_3 can be rewritten as

$$\begin{aligned}
\sigma_3(\zeta,\psi) &= -\delta_{x'}\left(\overline{(\delta_x v - \delta_y u)}^{x'} u'\right) - \delta_{y'}\left(\overline{(\delta_x v - \delta_y u)}^{y'} v'\right)\\
&= -\delta_{x'}\delta_x\left(\underset{\textcircled{1}}{\bar{v}^{x'}}\,\overline{u'}^{x}\right) + \delta_{x'}\left(\overline{\underset{\textcircled{2}}{\bar{v}^{x'}}\delta_x u'}^{x}\right)\\
&\quad - \delta_{y'}\delta_x\left(\underset{\textcircled{3}}{\bar{v}^{y'}}\,\overline{v'}^{x}\right) + \delta_{y'}\left(\overline{\underset{\textcircled{4}}{\bar{v}^{y'}}\delta_x v'}^{x}\right)\\
&\quad + \delta_{x'}\delta_y\left(\underset{\textcircled{5}}{\bar{u}^{x'}}\,\overline{v'}^{y}\right) - \delta_{x'}\left(\overline{\underset{\textcircled{6}}{\bar{u}^{x'}}\delta_y u'}^{y}\right)\\
&\quad + \delta_{y'}\delta_y\left(\underset{\textcircled{7}}{\bar{v}^{y'}}\,\overline{v'}^{y}\right) - \delta_{y'}\left(\overline{\underset{\textcircled{8}}{\bar{u}^{y'}}\delta_y v'}^{y}\right)
\end{aligned}$$

$\delta_x, \delta_y, \delta_{x'}, \delta_{y'}$ are all commutative →

① and ③ are FD analogs of $-\frac{\partial}{\partial x}\nabla\cdot(\vec{v}v)$

⑤ and ⑦ ———————— $-\frac{\partial}{\partial y}\nabla\cdot(\vec{v}u)$

Non divergent case $\delta_x u + \delta_y v = 0$

$$(25)\quad \begin{cases} u' = -\delta_{y'}\psi = \dfrac{\bar{u}^{x} + \bar{v}^{y}}{\sqrt{2}} \\ v' = \delta_{x'}\psi = -\dfrac{\bar{u}^{x} + \bar{v}^{y}}{\sqrt{2}} \end{cases} \qquad \left(\frac{1}{\sqrt{2}}\left(\delta_x\bar{\psi}^{y} - \delta_y\bar{\psi}^{x}\right)\right)$$

We then also have

$$\delta_x u' = -\delta_{y'} v$$
$$\delta_x v' = +\delta_{x'} v$$
$$\delta_x u' = \delta_{y'} u$$
$$\delta_y v' = -\delta_{x'} u$$

$\Rightarrow$ σ_3 can then be expressed as

$$\sigma_3(\zeta,\psi) = -\delta_x\left(\delta_{x'}(\overline{u'}^{x}\,\overline{v}^{x'}) + \delta_{y'}(\overline{v'}^{x}\,\overline{v}^{y'})\right.$$
$$\left. + \delta_y\left(\delta_{x'}(\overline{u'}^{y}\,\overline{u}^{x'}) + \delta_{y'}(\overline{v'}^{y}\,\overline{v}^{y'})\right)\right)$$
$$\neq \underbrace{\overline{\sigma_3(v,v)}^{x}}_{\text{not} = 0} \neq \underbrace{\overline{\sigma_3(-u,u)}^{y}}_{\text{not} = 0}$$

<u>No success</u>.

If we now try to express also $\frac{\sigma_1+\sigma_2}{2}$ with identifiable aspects of $\nabla\cdot(\vec{v}u)$ and $\nabla\cdot(\vec{v}v)$, we can come up with a workable $\sigma_A = (\sigma_1+\sigma_2+\sigma_3)/3$

$$\frac{\sigma_1+\sigma_2}{2}(\zeta,\psi) = -\delta_x(\overline{u}^{xy}\,\delta_x\overline{v}^{x}) + \delta_y(\overline{v}^{xy}\,\delta_x\overline{v}^{y})$$
$$+ \delta_x(\overline{u}^{xy}\,\delta_y\overline{v}^{x}) + \delta_y(\overline{v}^{xy}\,\delta_y\overline{v}^{y})$$
$$= -\delta_x\left(\delta_x(\overline{u}^{xxy}\,\overline{v}^{x}) + \delta_y(\overline{v}^{xxy}\,\overline{v}^{y})\right.$$
$$\left. + \delta_y\left(\delta_x(\overline{u}^{xyy}\,\overline{u}^{x}) + \delta_y(\overline{v}^{xyy}\,\overline{u}^{y})\right)\right)$$
$$- \overline{\frac{\sigma_1+\sigma_2}{2}(v,v)}^{x} - \overline{\frac{\sigma_1+\sigma_2}{2}(-u,u)}^{y}.$$

Since $\sigma_A(u,u) = \sigma_A(v,v) = 0$, we combine the two expressions and we obtain for σ_A

$$\sigma_A(\zeta,\psi) =$$

$$(26)\quad \begin{aligned} & - \delta_x \Big(\tfrac{2}{3}\Big(\delta_x(\overline{u}^{xxy}\, \overbrace{\overline{v}^{x}) + \delta_y(\overline{v}^{xxy}}^{\nabla\cdot(\vec{V}v)}\, \overline{v}^{y})\Big] \\ & \qquad + \tfrac{1}{3}\Big(\delta_{x'}(\overline{u'}^{x}\, \overline{v}^{x'}) + \delta_{y'}(\overline{v'}^{x}\, \overline{v}^{y'})\Big)\Big]\Big) \\ & + \delta_y \Big(\tfrac{2}{3}\Big(\delta_x(\overline{u}^{xyy}\, \overline{u}^{x}) + \delta_y(\overline{v}^{xyy}\, \overline{u}^{y})\Big)\Big] \\ & \qquad + \tfrac{1}{3}\Big[\underbrace{\delta_{x'}(\overline{u'}^{y}\, \overline{u}^{x'}) + \delta_{y'}(\overline{v'}^{y}\, \overline{u}^{y'})}_{\nabla\cdot(\vec{V}u)}\Big)\Big]\Big]\Big) \end{aligned}$$

These FD operators for $\nabla\cdot(\vec{V}u)$ and $\nabla\cdot(\vec{V}v)$ in this expression guarantee conservation of both energy and enstrophy by the non-divergent part of the flow. To obtain momentum advection operator applicable to divergent flow, suitable FD analogs of $-u\,\nabla\cdot\vec{V}$ and $-v\,\nabla\cdot\vec{V}$ must be added to the operators. (See Arakawa and Lamb, 1977 for details)

Another way of adapting the above operator to divergent flow conditions is to transform the equations of motions into flux form. For example, in the case of the shallow-water equations,

$$\left\{ \begin{aligned} & \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial h}{\partial x} \\ & \frac{\partial v}{\partial t} \mp u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -g\frac{\partial h}{\partial y} \\ & \frac{\partial h}{\partial t} + \nabla(h\vec{V}) = 0 \end{aligned} \right.$$

& $hu\frac{\partial u}{\partial x} + hv\frac{\partial u}{\partial y}$

$$h\frac{\partial u}{\partial t} + hu\frac{\partial u}{\partial x} + hv\frac{\partial v}{\partial y} = \frac{\partial(hu)}{\partial t} \underbrace{-u\frac{\partial h}{\partial t}}_{+u\frac{\partial hu}{\partial x} + u\frac{\partial hv}{\partial y}} \quad h\frac{\partial(\cancel{hu})}{} + \nabla(\cancel{hu}) \quad + hu\frac{\partial u}{\partial x} + \partial h\frac{\partial u}{\partial x} = -gh\frac{\partial h}{\partial x}$$

$$\frac{\partial}{\partial x} huu + \frac{\partial}{\partial y}(hvu) \quad \underline{\qquad} \quad \underline{\quad}.$$

The horizontal advection terms then assume the form (for $\frac{\partial (hu)}{\partial t} = \dots \quad \frac{\partial (hv)}{\partial t} = \dots$)

$$\nabla . \left((h\vec{v}) u \right) \quad \text{and} \quad \nabla \cdot \left((h\vec{v}) v \right)$$

If U and V are the components of $h\vec{v}$, the momentum flux operators can then be written making use of (25)

$$\nabla \cdot \left((h\vec{v}) u \right) = \frac{2}{3} \left[\delta_x \left(\overline{U}^{xyy} \, \overline{u}^{x} \right) + \delta_y \left(\overline{V}^{xyy} \, \overline{u}^{y} \right) \right]$$

$$+ \frac{1}{3} \left[\delta_{x'} \left(\frac{\overline{U}^{xy} + \overline{V}^{yy}}{\sqrt{2}} \, \overline{u}^{x'} \right) + \delta_{y'} \left(\frac{-\overline{U}^{xy} + \overline{V}^{yy}}{\sqrt{2}} \, \overline{u}^{y'} \right) \right]$$

$$\nabla \cdot \left((h\vec{v}) v \right) = \frac{2}{3} \left[\delta_x \left(\overline{U}^{xyy} \, \overline{v}^{x} \right) + \delta_y \left(\overline{V}^{xyy} \, \overline{v}^{y} \right) \right]$$

$$+ \frac{1}{3} \left(\delta_{x'} \left(\frac{\overline{U}^{xx} + \overline{V}^{xy}}{\sqrt{2}} \, \overline{v}^{x'} \right) + \delta_{y'} \left(\frac{-\overline{U}^{xx} + \overline{V}^{xy}}{\sqrt{2}} \, \overline{v}^{y'} \right) \right]$$

5.5. Potential Vorticity conservation in the momentum equations. (Sadourny, 1975)

So far, we have mostly dealt with advection operators which handle the conservation laws for the non-divergent part of the velocity field, which is often dominant in rotating fluids. There are situations such as flows over obstacles where the divergent velocity plays a significant role. Under such conditions, it may be important to incorporate conservation laws applicable to non divergent flow in the FD operators. One of the most important is conservation of potential vorticity

Let's consider the shallow water case

$$\begin{cases} \dfrac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} + f\,\vec{k}\times\vec{v} = -g\nabla(h+h_s) & (26) \\[2ex] \dfrac{\partial h}{\partial t} + \nabla\cdot(h\vec{v}) = 0 & (27) \end{cases}$$

The potential vorticity equation is derived by

1) expressing $\vec{v}.\nabla\vec{v}$ [illegible] $\nabla\left(\vec{v}^2/2\right) - \vec{v}\times\zeta\,\vec{k}$

$\left(\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

2) apply $\nabla\times$ to (26)

$\Rightarrow$ the vorticity equation

$$\boxed{\frac{\partial}{\partial t}(\zeta+f) + \nabla\cdot\left((\zeta+f)\vec{v}\right) = 0} \qquad (28)$$

using (27) gives the potential vorticity eqn:

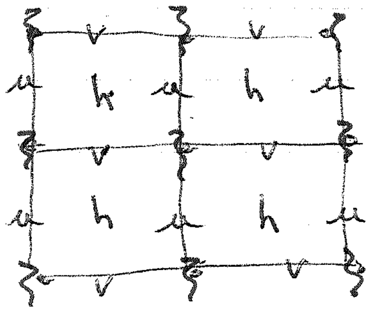
$$\boxed{\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)\left(\frac{\zeta+f}{h}\right) = 0} \qquad (29)$$

(29) conserves the volume integral of the potential vorticity $\iint\left(\frac{\zeta+f}{h}\right)h\,dx\,dy$ because of (28).

The vorticity equation is the flux form of the potential vorticity equation.

From the point of view of solving the continuity equation (27), the best grid is the "C" grid

ζ = vorticity $= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$



The continuity equation then assumes the simple form

(30) $$\frac{\partial h}{\partial t} + \delta_x U + \delta_x V = 0$$

with $U = \overline{h}^{x} u \qquad V = \overline{h}^{y} v$

Vorticity ζ and potential vorticity q are carried out at the vorticity points

(31) $$\left\{ \begin{aligned} \zeta &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \delta_x v - \delta_y u \\ q &= \frac{\delta_x v - \delta_y u + f}{\overline{h}^{xy}} \end{aligned} \right.$$

The purpose now is to formulate FD expressions for the momentum advection terms compatible with (29)

With this in mind, we start from (30) and multiply by q after averaging (30) with $\overline{}^{xy}$ to bring the points on q location

$$q \frac{\partial \overline{h}^{xy}}{\partial t} + q\left(\overline{\delta_x U}^{xy} + \overline{\delta_x V}^{xy}\right) = 0$$

$$= \frac{\partial(\overline{h}^{xy} q)}{\partial t} + \delta_x\left(\overline{q}^{x}\, \overline{U}^{xy}\right) + \delta_y\left(\overline{q}^{y}\, \overline{V}^{xy}\right)$$

$$- \overline{h}^{xy} \frac{\partial q}{\partial t} - \overline{\overline{U}^{xy} \delta_x q}^{x} - \overline{\overline{V}^{xy} \delta_y q}^{y} = 0$$

which can be rewritten as

(32) $$\frac{\partial(\overline{h}^{xy} q)}{\partial t} + \delta_x\left(\overline{q}^{x}\, \overline{U}^{xy}\right) + \delta_y\left(\overline{q}^{y} \cdot \overline{V}^{xy}\right)$$

$$= \overline{h}^{xy}\left(\frac{\partial q}{\partial t} + \frac{1}{\overline{h}^{xy}}\left(\overline{\overline{U}^{xy} \delta_x q}^{x} + \overline{\overline{V}^{xy} \delta_y q}^{y}\right)\right)$$

This identity is an FD expression of

$$\frac{\partial}{\partial t}(qh) + \nabla\cdot(qh\vec{v}) = h\frac{dq}{dt}$$

If we choose the RHS of (32) as being the FD expression for the potential vorticity eqn (29), the LHS is an FD analog to the vorticity eqn (28)

Carrying the steps backwards

$$\underline{-\vec{v}\times(\zeta+f)\vec{k}} = \begin{pmatrix} -v(\zeta+f) \\ +u(\zeta+f) \end{pmatrix}$$

from (28) and (32) $\Rightarrow$ FD form $\begin{pmatrix} -\overline{V}^{xy}\,\overline{q}^{y} \\ \overline{U}^{xy}\,\overline{q}^{x} \end{pmatrix}$

Combined with a suitable form of $\nabla\frac{\vec{v}^2}{2}$, we should arrive to the desired potential vorticity conserving operator for the advection and Coriolis term $\vec{v}\cdot\nabla\vec{v} + f\vec{k}\times\vec{v}$

We have 2 choices $\begin{cases} \frac{1}{2}\delta_x\left((\overline{u}^x)^2 + (\overline{v}^y)^2\right) \\ \frac{1}{2}\delta_y\left((\overline{u}^x)^2 + (\overline{v}^y)^2\right) \end{cases}$

or $\begin{cases} \frac{1}{2}\delta_x\left(\overline{u^2}^x + \overline{v^2}^y\right) \\ \frac{1}{2}\delta_y\left(\overline{u^2}^x + \overline{v^2}^y\right) \end{cases}$

Both satisfy the requirement that they drop out upon cross differentiation. Choice (2) however leads to a straightforward FD analog of the kinetic energy and is therefore preferred

This leads to the following potential vorticity conserving momentum advection and Coriolis force operators

(33) $$\begin{cases} \vec{v}\cdot\nabla u - f v = \frac{1}{2}\delta_x\left(\overline{u^2}^{x} + \overline{v^2}^{y}\right) - \overline{V}^{xy}\,\overline{q}^{y} \\ \vec{v}\cdot\nabla v + f u = \frac{1}{2}\delta_y\left(\overline{u^2}^{x} + \overline{v^2}^{y}\right) + \overline{U}^{xy}\,\overline{q}^{x} \end{cases}$$

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this actually corresponds to a modification of the enstrophy conserving Jacobian $\left(\frac{J_1 + J_2}{2}\right)$ with the substitution $(u, v) \Leftrightarrow U, V$

$$q \Longleftrightarrow \zeta$$

$$\left(-\frac{J_1+J_2}{2}(\zeta,\psi) = \delta_x\left(\underline{\overline{u}^{xy}\,\overline{\zeta}^{x}}\right) + \delta_y\left(\underline{\overline{v}^{xy}\,\overline{\zeta}^{y}}\right)\right).$$

(The cross-differentiation makes the energy term to disappear).

Since $(J_1+J_2)/2$ conserves $\zeta^2/2$, we may expect (33) to conserve $q^2/2$, potential enstrophy

Indeed if we multiply (32) by q, we obtain

(34) $$q^2\frac{\partial \overline{h}^{xy}}{\partial t} + \overline{h}^{xy}\frac{\partial}{\partial t}\left(q^2/2\right) + \delta_x\left(\overline{U}^{xy}\,\widetilde{\frac{q^2}{2}}^{x}\right) + \delta_y\left(\overline{V}^{xy}\,\widetilde{\frac{q^2}{2}}^{y}\right) + \frac{q^2}{2}\left(\overline{\delta_x U + \delta_y V}\right)^{xy} = 0$$

Using the continuity eqn. (28), the last term can be combined with the $\frac{\partial}{\partial t}$ terms to give

$$\frac{\partial}{\partial t}\left(\overline{h}^{xy}q^2/2\right)$$

$\Longrightarrow$ that the FD (34) is an FD analog to

(35) $$\boxed{\frac{\partial}{\partial t}\left(h\,q^2/2\right) + \nabla\cdot\left(h\vec{v}\,q^2/2\right) = 0}$$ Conservation of potential enstrophy

Parallels to other Jacobians can be made, but generalisation to an Arakawa Jacobian is not straightforward, primarily because kinetic energy consideration cannot be based on arguments involving the streamfunction ψ.

Arakawa and Lamb (1981) did derive a scheme which conserves both energy and potential enstrophy. The derivation is very complex and leads to the following operators.

$$\left\{\begin{aligned}
&\tfrac{1}{2}\,\delta_x\left(\overline{u^2}^x + \overline{v^2}^y\right) - \overline{\overline{v}^y\,\overline{q}^{xy}}^x - \tfrac{1}{48}\,\delta'_x\left((\delta'_y v)\,\delta'_x\delta'_y q\right)\\
&\qquad + \tfrac{1}{12}\,\delta'_x\left(\overline{u}^x\,\delta'_y\,\overline{q}^x\right) + \tfrac{1}{12}\,\overline{\left(\delta'_x u\;\delta'_y\,\overline{q}^x\right)}^x\\[2ex]
&\tfrac{1}{2}\,\delta_y\left(\overline{u^2}^x + \overline{v^2}^y\right) + \overline{\overline{u}^x\,\overline{q}^{xy}}^y + \tfrac{1}{48}\,\delta'_y\left((\delta'_x u)\,\delta'_x\delta'_y q\right)\\
&\qquad - \tfrac{1}{12}\,\delta'_y\left(\overline{v}^y\,\delta_x\,\overline{q}^y\right) - \tfrac{1}{12}\,\overline{\left(\delta'_y v\;\delta'_x\,\overline{q}^y\right)}^y
\end{aligned}\right.$$

where δ' is the differential operator without division by grid distance