# A Potential Enstrophy and Energy Conserving Scheme for the Shallow Water Equations

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ABSTRACT

To improve the simulation of nonlinear aspects of the flow over steep topography, a potential enstrophy and energy conserving scheme for the shallow water equations is derived. It is pointed out that a family of schemes can conserve total energy for general flow and potential enstrophy for flow with no mass flux divergence. The newly derived scheme is a unique member of this family, that conserves both potential enstrophy and energy for general flow. Comparison by means of numerical experiment with a scheme that conserves (potential) enstrophy for purely horizontal nondivergent flow demonstrated the considerable superiority of the newly derived potential enstrophy and energy conserving scheme, not only in suppressing a spurious energy cascade but also in determining the overall flow regime. The potential enstrophy and energy conserving scheme for a spherical grid is also presented.

## 1. Introduction

The possibility of improvement in the prediction of planetary-scale waves, which should be relatively free of truncation errors in a linearized system, rests on evidence that the mechanisms for their generation and subsequent time change are nonlinear and involve smaller scales. Indeed, medium-range numerical prediction experiments (e.g., Miyakoda *et al.*, 1977) have shown that a decrease in horizontal grid size improved the predictability even of large, and presumably already well-resolved, planetary waves. Thus, improved prediction of these waves with conventional GCM grid sizes must be accomplished by better simulation of the underlying nonlinear mechanisms.

In nature, steep mountains play an extremely important role in the generation and maintenance of both planetary- and cyclone-scale waves. Cyclone activities are by no means uniform in longitude and are subject, not to the zonal mean baroclinicity, but to local baroclinicity produced, in part, by the earth's large-scale topography. Lee cyclones are produced that interact with planetary waves, not only dynamically but also thermodynamically through forcing due to the release of latent heat. In addition, steep mountains can directly influence the dynamics of planetary waves, even if they are longitudinally very narrow, provided that they are sufficiently high and have a large enough meridional extent. When conventional GCM grid sizes are used, however, even the Rocky Mountains extend at most only a few grid intervals in longitude and, therefore, care must be taken to reduce the possibility of serious truncation error in the dynamical response of the model atmosphere.

The effect of truncation error is minimized for a given differencing scheme when the grid size is in a range for which the solution shows no significant change with increased resolution. However, whether a given grid size is in such a range or, indeed, whether such a range can be found is highly scheme-dependent in nonlinear systems. For example, Arakawa and Lamb (1977) showed that some common schemes for two-dimensional incompressible flow produce a spurious energy cascade even though the total energy is conserved. After time integrations of sufficient length with such schemes, a significant amount of energy exists in the smallest resolvable scales, where truncation error is large. Under such conditions a decrease in the grid size will always affect the solution. On the other hand, solutions with a scheme that prevents a false energy cascade should be relatively smooth and, therefore, should not be significantly affected by a decrease in the grid size. This means that the original solution is already a good approximation. Such a scheme can be found only by making the nonlinear aspects of the dynamics of the discrete system as close as possible to those of the original continuous system.

It should be pointed out that increasing the order of accuracy does not necessarily guarantee much improvement. If a significant amount of energy exists in the smallest resolvable scales resulting from a spurious energy cascade, convergence of the Taylor expansion of the truncation error may not be

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sufficiently rapid due to large values of higher order derivatives.

In this paper a differencing scheme is sought whose dynamics represent well even the nonlinear aspects of the flow over steep topography of a homogeneous incompressible shallow fluid. In such a fluid, flow over and near mountains is governed during advective processes by the conservation of (absolute) potential vorticity $\eta/h$, where $\eta$ is the (absolute) vorticity and $h$ is the depth of the fluid. Consequently, the (absolute) potential enstrophy $\overline{\tfrac{1}{2}\eta^2/h}$ is conserved, where the overbar means a horizontal average. Since

$$\overline{\eta^2} < \overline{\frac{h_{\max}}{h}\eta^2} = h_{\max}(\overline{\eta^2/h}) = \text{constant},$$

there is an upper bound for (absolute) enstrophy $\overline{\tfrac{1}{2}\eta^2}$. Therefore, in this system an energy cascade is restricted though $h$ is variable, as it is in a purely two-dimensional flow.

We have found that conventional space finite-difference schemes for the momentum equation, when applied to the shallow water equations, correspond to very bad advection schemes for the potential vorticity in the presence of steep mountains. In particular, conservation of potential enstrophy is not guaranteed, even when the scheme guarantees enstrophy conservation for a purely two-dimensional flow. To overcome this deficiency, a space finite-difference scheme for the shallow water momentum equations was designed to conserve potential enstrophy as well as total energy.

In Section 2, the shallow water equations are presented and a method of derivation of potential enstrophy conserving schemes is outlined. The method is used in Section 3 to derive a scheme for which conservation of potential enstrophy is guaranteed in the general case of divergent mass flux. In Section 4, a family of schemes that conserve potential enstrophy only in the special case of nondivergent mass flux is derived. The advantages of the potential enstrophy conserving scheme derived in Section 3 are demonstrated in Section 5 through a comparison, by means of numerical time integrations, with a scheme that conserves (potential) enstrophy only for purely two-dimensional flow. The Appendix presents the potential enstrophy and energy conserving scheme for a spherical grid that can be derived by analogy to the procedure in Section 3.

## 2. Outline of the derivation procedure

The governing differential equations for quasi-static motion in a homogeneous incompressible fluid with a free surface can be written as

$$\frac{\partial \mathbf{v}}{\partial t} + q\mathbf{k} \times \mathbf{v}^* + \nabla(K + \Phi) = 0, \tag{2.1}$$

$$\frac{\partial h}{\partial t} + \nabla \cdot \mathbf{v}^* = 0. \tag{2.2}$$

Here the (absolute) potential vorticity $q$ and the mass flux $\mathbf{v}^*$ are defined by

$$\left.\begin{aligned} q &\equiv (f + \zeta)h^{-1} \\ \mathbf{v}^* &\equiv h\mathbf{v} \end{aligned}\right\}, \tag{2.3}$$

and $\mathbf{v}$ is the horizontal velocity, $t$ the time, $f$ the Coriolis parameter, $\zeta$ the vorticity, $\mathbf{k}\cdot\nabla \times \mathbf{v}$, $\mathbf{k}$ the vertical unit vector, $\nabla$ the horizontal del operator, $h$ the vertical extent of a fluid column above the bottom surface, $K$ the kinetic energy per unit mass, $\tfrac{1}{2}\mathbf{v}^2$, $g$ the gravitational acceleration, $h_s$ the bottom surface height, and

$$\Phi \equiv g(h + h_s). \tag{2.5}$$

Multiplying (2.1) by $\mathbf{v}^*$ and combining the results with (2.2) yield the equation for the time change of total kinetic energy

$$\frac{\partial}{\partial t}(hK) + \nabla\cdot(\mathbf{v}^*K) + \mathbf{v}^*\cdot\nabla\Phi = 0. \tag{2.6}$$

Multiplying (2.2) by $\Phi$ gives the equation for the time change of potential energy,

$$\frac{\partial}{\partial t}(\tfrac{1}{2}gh^2 + ghh_s) + \nabla\cdot(\mathbf{v}^*\Phi) - \mathbf{v}^*\cdot\nabla\Phi = 0. \tag{2.7}$$

The summation of (2.6) and (2.7) then yields a statement of the conservation of total energy

$$\frac{\partial}{\partial t}\overline{[h(K + \tfrac{1}{2}gh + gh_s)]} = 0, \tag{2.8}$$

where the overbar, here and in the text to follow, denotes the mean over an infinite domain or over a finite domain with no inflow or outflow through the boundaries. Obviously, the term in (2.1) involving $q$ makes no contribution to the change of total kinetic energy. Also, the last term in (2.6) and (2.7) cancel in giving (2.8). These points will be used in the construction of the finite-difference scheme.

The vorticity equation for this fluid motion is obtained from (2.1) and may be written in the form

$$\frac{\partial}{\partial t}(hq) + \nabla\cdot(\mathbf{v}^*q) = 0. \tag{2.9}$$

Subtracting (2.2) times $q$ from (2.9) and dividing by $h$ gives

$$\frac{\partial q}{\partial t} + \mathbf{v}\cdot\nabla q = 0. \tag{2.10}$$

Thus potential vorticity is simply advected and in the absence of spatial gradients of $q$ there should be no time change of $q$. Only schemes that preserve this property will be considered in this paper.

Now $hq$ times (2.10) plus $\tfrac{1}{2}q^2$ times (2.2) gives the equation for time change of potential enstrophy

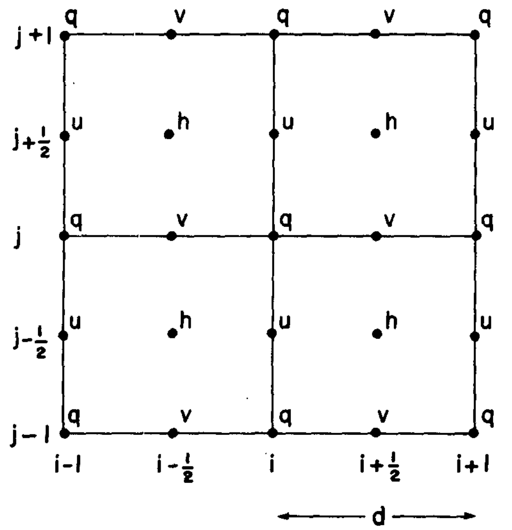


FIG. 1. The staggering of the variables based on the C grid to be used in the derivation of the square grid version of the potential enstrophy and energy conserving scheme.

$$\frac{\partial}{\partial t}(h\tfrac{1}{2}q^2) + \nabla\cdot(\mathbf{v}^*\tfrac{1}{2}q^2) = 0, \tag{2.11}$$

which leads to a statement of the conservation of potential enstrophy,

$$\frac{\partial}{\partial t}(\overline{h\tfrac{1}{2}q^2}) = 0. \tag{2.12}$$

Since our goal is the derivation of a finite-difference scheme for the momentum equations that better represents the flow over steep bottom topography, the first requirement we impose is that it be consistent with a reasonable advection scheme for potential vorticity advection equation (2.10). In particular, because the scheme is to be used in a long-term integration, we require conservation of total energy and potential enstrophy, as given by (2.8) and (2.12), respectively. The derivation procedure for such a scheme is outlined here and presented in detail in the next section.

A general difference scheme for (2.1) can be written to directly guarantee conservation of total kinetic energy in the special case of nondivergent mass flux. Even after the constraints necessary to achieve total energy conservation in the divergent mass flux case are applied, the scheme still retains a high degree of freedom. We first require that when $q$ is constant in space, there is no time change of $q$. Then, to guarantee conservation of potential enstrophy, we require that the finite-difference analog of (2.12) holds. These requirements essentially fix the scheme; the small remaining freedom is used to preserve symmetry between the Cartesian components of the momentum equations for the case of a square grid.

## 3. Derivation of the potential enstrophy conserving scheme

The staggering of the variables to be used in this derivation, called the C grid, is shown in Fig. 1. Here $u$ and $v$ are the Cartesian components of $\mathbf{v}$ in $x$ and $y$ directions, respectively. The choice of the C grid is based on the fact that it best simulates the geostrophic adjustment mechanism (Arakawa and Lamb, 1977). The indexing will be as shown in the figure, with the indices $(i,j)$ used for the vorticity points. The time derivatives will be left, for simplicity, in differential form throughout.

The differencing for the continuity equation (2.2) can be written

$$\frac{\partial}{\partial t} h_{i+1/2,j+1/2} + (\nabla\cdot\mathbf{v}^*)_{i+1/2,j+1/2} = 0, \tag{3.1}$$

where

$$(\nabla\cdot\mathbf{v}^*)_{i+1/2,j+1/2} \equiv \frac{1}{d}[u^*_{i+1,j+1/2} - u^*_{i,j+1/2} + v^*_{i+1/2,j+1} - v^*_{i+1/2,j}], \tag{3.2}$$

$$u^*_{i+1,j+1/2} \equiv [h^{(u)}u]_{i+1,j+1/2}, \tag{3.3}$$

$$v^*_{i+1/2,j} \equiv [h^{(v)}v]_{i+1/2,j}, \tag{3.4}$$

and $h^{(u)}$ and $h^{(v)}$ are the $h$ values at $u$ and $v$ points, respectively, as yet unspecified.

The general second-order scheme chosen to represent the Cartesian components of the momentum equation (2.1) is

$$\begin{aligned}\frac{\partial}{\partial t} u_{i,j+1/2} &- \alpha_{i,j+1/2}v^*_{i+1/2,j+1} - \beta_{i,j+1/2}v^*_{i-1/2,j+1} \\ &- \gamma_{i,j+1/2}v^*_{i-1/2,j} - \delta_{i,j+1/2}v^*_{i+1/2,j} + \epsilon_{i+1/2,j+1/2}u^*_{i+1,j+1/2} \\ &- \epsilon_{i-1/2,j+1/2}u^*_{i-1,j+1/2} + d^{-1}[(K+\Phi)_{i+1/2,j+1/2} \\ &- (K+\Phi)_{i-1/2,j+1/2}] = 0,\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\frac{\partial}{\partial t} v_{i+1/2,j} &+ \gamma_{i+1,j+1/2}u^*_{i+1,j+1/2} + \delta_{i,j+1/2}u^*_{i,j+1/2} \\ &+ \alpha_{i,j-1/2}u^*_{i,j-1/2} + \beta_{i+1,j-1/2}u^*_{i+1,j-1/2} \\ &+ \phi_{i+1/2,j+1/2}v^*_{i+1/2,j+1} - \phi_{i+1/2,j-1/2}v^*_{i+1/2,j-1} \\ &+ d^{-1}[(K+\Phi)_{i+1/2,j+1/2} \\ &- (K+\Phi)_{i+1/2,j-1/2}] = 0.\end{aligned} \tag{3.6}$$

The symbols $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$ and $\phi$ are linear combinations of the $q$ and $K$ is defined at the $h$ points; the actual forms of all are as yet unspecified. The pairs of terms involving $\epsilon$ and $\phi$, which give additional generality to the scheme, should vanish when the grid size approaches zero as required for consistency.

Multiplying (3.5) by $u^*_{i,j+1/2}$ and (3.6) by $v^*_{i+1/2,j}$ and summing the resulting equations over the entire domain allows us to write an equation for the time change of total kinetic energy:

$$\sum_{u\text{ pts}} \frac{\partial}{\partial t}(h^{(u)}\tfrac{1}{2}u^2)_{i,j+1/2} + \sum_{v\text{ pts}} \frac{\partial}{\partial t}(h^{(v)}\tfrac{1}{2}v^2)_{i+1/2,j} - \sum_{u\text{ pts}}\left(\tfrac{1}{2}u^2\frac{\partial}{\partial t}h^{(u)}\right)_{i,j+1/2} - \sum_{v\text{ pts}}\left(\tfrac{1}{2}v^2\frac{\partial}{\partial t}h^{(v)}\right)_{i+1/2,j} - \sum_{h\text{ pts}}[(K+\Phi)\nabla\cdot\mathbf{v}^*]_{i+1/2,j+1/2} = 0. \quad (3.7)$$

Here we have made use of the fact that for any variables $a$, $b$ defined at staggered points on the grid,

$$\sum_{a\text{ pts}} a_{i,j}(b_{i+1/2,j} - b_{i-1/2,j}) = -\sum_{b\text{ pts}} b_{i+1/2,j}(a_{i+1,j} - a_{i,j}), \quad (3.8)$$

and a similar relation with regard to the $j$ index. Note that the terms involving $q$ in the momentum equations have cancelled simply by virtue of the form adopted in (3.5) and (3.6).

In the special case of nondivergent mass flux, for which $\nabla\cdot\mathbf{v}^* = 0$, Eq. (3.1) reduces to $\partial h_{i+1/2,j+1/2}/\partial t = 0$, and the total kinetic energy as given in the first two terms of (3.7) is conserved regardless of the forms of $h^{(u)}$, $h^{(v)}$ or $K$. To maintain conservation of the same form of total kinetic energy in the case for which $\nabla\cdot\mathbf{v}^* \neq 0$ (but without the pressure gradient forces), it is necessary that $h^{(u)}$, $h^{(v)}$ and $K$ be chosen such that through the use of (3.1) we can write

$$\sum_{u\text{ pts}}\left(\tfrac{1}{2}u^2\frac{\partial}{\partial t}h^{(u)}\right)_{i,j+1/2} + \sum_{v\text{ pts}}\left(\tfrac{1}{2}v^2\frac{\partial}{\partial t}h^{(v)}\right)_{i+1/2,j} + \sum_{h\text{ pts}}(K\nabla\cdot\mathbf{v}^*)_{i+1/2,j+1/2} = 0. \quad (3.9)$$

For the moment, however, $h^{(u)}$, $h^{(v)}$ and thus $K$ will be left unspecified because the potential enstrophy conservation constraint to be applied does not depend on the form chosen for these variables.

Application of (3.5) and (3.6) at the points surrounding a $\zeta$ point gives the finite-difference vorticity equation consistent with this scheme:

$$\begin{aligned}\frac{\partial}{\partial t}(h^{(q)}q)_{i,j} &= d^{-1}[-v^*_{i+1/2,j+1}(\alpha_{i,j+1/2} + \phi_{i+1/2,j+1/2}) - v^*_{i-1/2,j+1}(\beta_{i,j+1/2} - \phi_{i-1/2,j+1/2}) + v^*_{i+1/2,j}(\alpha_{i,j-1/2} - \delta_{i,j+1/2})\\ &+ v^*_{i-1/2,j}(\beta_{i,j-1/2} - \gamma_{i,j+1/2}) + v^*_{i+1/2,j-1}(\delta_{i,j-1/2} + \phi_{i+1/2,j-1/2}) + v^*_{i-1/2,j-1}(\gamma_{i,j-1/2} - \phi_{i-1/2,j-1/2})\\ &- u^*_{i+1,j+1/2}(\gamma_{i+1,j+1/2} - \epsilon_{i+1/2,j+1/2}) - u^*_{i,j+1/2}(\delta_{i,j+1/2} - \gamma_{i,j+1/2}) + u^*_{i-1,j+1/2}(\delta_{i-1,j+1/2} - \epsilon_{i-1/2,j+1/2})\\ &- u^*_{i+1,j-1/2}(\beta_{i+1,j-1/2} + \epsilon_{i+1/2,j-1/2}) - u^*_{i,j-1/2}(\alpha_{i,j-1/2} - \beta_{i,j-1/2}) + u^*_{i-1,j-1/2}(\alpha_{i-1,j-1/2} + \epsilon_{i-1/2,j-1/2})],\end{aligned} \quad (3.10)$$

where the vorticity change has been expressed as $\partial(h^{(q)}q)_{i,j}/\partial t$, with

$$q_{ij} \equiv \frac{(f+\zeta)_{i,j}}{h^{(q)}{}_{i,j}}, \quad (3.11)$$

$$\zeta_{i,j} \equiv d^{-1}[u_{i,j-1/2} - u_{i,j+1/2} + v_{i+1/2,j} - v_{i-1/2,j}], \quad (3.12)$$

and $h^{(q)}$ is a linear combination of $h$, as yet unspecified.

We first require that $\partial q/\partial t$ vanish when $q$ is formally set equal to constant on the right-hand side of (3.10), regardless of the constant. If we write $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$ and $\phi$ in general form as linear combinations of the surrounding $q$,

$$\left.\begin{aligned}\alpha_{i,j+1/2} &= \alpha^{(1)}q_{i+1,j+1} + \alpha^{(2)}q_{i,j+1} + \alpha^{(3)}q_{i,j} + \alpha^{(4)}q_{i+1,j}\\ \beta_{i,j+1/2} &= \beta^{(1)}q_{i,j+1} + \beta^{(2)}q_{i-1,j+1} + \beta^{(3)}q_{i-1,j} + \beta^{(4)}q_{i,j}\\ \gamma_{i,j+1/2} &= \gamma^{(1)}q_{i,j+1} + \gamma^{(2)}q_{i-1,j+1} + \gamma^{(3)}q_{i-1,j} + \gamma^{(4)}q_{i,j}\\ \delta_{i,j+1/2} &= \delta^{(1)}q_{i+1,j+1} + \delta^{(2)}q_{i,j+1} + \delta^{(3)}q_{i,j} + \delta^{(4)}q_{i+1,j}\\ \epsilon_{i+1/2,j+1/2} &= \epsilon^{(1)}q_{i+1,j+1} + \epsilon^{(2)}q_{i,j+1} + \epsilon^{(3)}q_{i,j} + \epsilon^{(4)}q_{i+1,j}\\ \phi_{i+1/2,j+1/2} &= \phi^{(1)}q_{i+1,j+1} + \phi^{(2)}q_{i,j+1} + \phi^{(3)}q_{i,j} + \phi^{(4)}q_{i+1,j}\end{aligned}\right\}, \quad (3.13)$$

then when $q$ is formally set equal to a constant, (3.10) can be written as

$$\begin{aligned}\frac{\partial}{\partial t}h^{(q)}_{i,j} &= -\frac{1}{d}[(A+F)(v^*_{i+1/2,j+1} - v^*_{i+1/2,j}) + (C-E)(u^*_{i+1,j+1/2} - u^*_{i,j+1/2}) + (B-F)(v^*_{i-1/2,j+1} - v^*_{i-1/2,j})\\ &+ (D-E)(u^*_{i,j+1/2} - u^*_{i-1,j+1/2}) + (C-F)(v^*_{i-1/2,j} - v^*_{i-1/2,j-1}) + (A+E)(u^*_{i,j-1/2} - u^*_{i-1,j-1/2})\\ &+ (D+F)(v^*_{i+1/2,j} - v^*_{i+1/2,j-1}) + (B+E)(u^*_{i+1,j-1/2} - u^*_{i,j-1/2})],\end{aligned} \quad (3.14)$$

where $A \equiv \sum_k \alpha^{(k)}, B \equiv \sum_k \beta^{(k)}$, etc. For any specification of $h^{(q)}$ this equation can only be satisfied if the values of $A$, $B$, $C$, $D$, $E$ and $F$ render it consistent with the continuity equation (3.1).

In the present case of a square grid, for simplicity and geometrical symmetry, $h^{(q)}$ is defined as

$$h^{(q)}_{ij} = \tfrac{1}{4}(h_{i+1/2,j+1/2} + h_{i-1/2,j+1/2} + h_{i-1/2,j-1/2} + h_{i+1/2,j-1/2}). \quad (3.15)$$

Now from (3.1) and (3.15) we can write

$$\frac{\partial}{\partial t} h^{(q)}_{i,j} = -(1/4d)[(v^*_{i+1/2,j+1} - v^*_{i+1/2,j} + u^*_{i+1,j+1/2} - u^*_{i,j+1/2}) + (v^*_{i-1/2,j+1} - v^*_{i-1/2,j} + u^*_{i,j+1/2} - u^*_{i-1,j+1/2}) + (v^*_{i-1/2,j} - v^*_{i-1/2,j-1} + u^*_{i,j-1/2} - u^*_{i-1,j-1/2}) + (v^*_{i+1/2,j} - v^*_{i+1/2,j-1} + u^*_{i-1,j-1/2} - u^*_{i,j-1/2})]. \quad (3.16)$$

Comparison of (3.14) and (3.16) yields the constraints

$$E = F = 0; \quad A = B = C = D = \tfrac{1}{4}. \quad (3.17)$$

The complete specification of $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$ and $\phi$ will be determined by requiring that the scheme conserve potential enstrophy in the general case of divergent mass flux. To formulate the necessary and sufficient conditions for this requirement, we can use the expressions (3.13) to formally rewrite (3.10) as

$$\frac{\partial}{\partial t}(h^{(q)}q)_{i,j} + \sum_{i',j'\neq 0} a_{i,j;i+i',j+j'}q_{i+i',j+j'} + b_{i,j}q_{i,j} = 0, \quad (3.18)$$

where the $a$ and $b$ are linear combinations of $u^*$ and $v^*$. We can then express (3.14) as

$$\frac{\partial}{\partial t} h^{(q)}_{i,j} + \sum_{i',j'\neq 0} a_{i,j;i+i',j+j'} + b_{i,j} = 0. \quad (3.19)$$

Subtracting (3.19) times $q_{i,j}$ from (3.18), multiplying the result by $q_{i,j}$ and adding it to $\tfrac{1}{2}q^2_{i,j}$ times (3.19) gives the desired potential enstrophy equation:

$$\frac{\partial}{\partial t}(h^{(q)}\tfrac{1}{2}q^2)_{i,j} + \sum_{i',j'\neq 0}(a_{i,j;i+i',j+j'}q_{i+i',j+j'}q_{i,j}) + \tfrac{1}{2}q^2_{i,j}(b_{i,j} - \sum_{i',j'\neq 0} a_{i,j;i+i',j+j'}) = 0. \quad (3.20)$$

Conservation of potential enstrophy over the domain, i.e.,

$$\sum_{i,j}\frac{\partial}{\partial t}\tfrac{1}{2}(h^{(q)}q^2)_{i,j} = 0 \quad (3.21)$$

is thus guaranteed if and only if

$$\sum_{i,j}\sum_{i',j'\neq 0} a_{i,j;i+i',j+j'}q_{i+i',j+j'}q_{i,j} = 0, \quad (3.22)$$

$$\sum_{i,j}\tfrac{1}{2}q^2_{i,j}[b_{i,j} - \sum_{i',j'\neq 0} a_{i,j;i+i',j+j'}] = 0. \quad (3.23)$$

To satisfy (3.22) we must have

$$a_{i,j;i+i',j+j'} = -a_{i+i';j+j',i,j}, \quad (3.24)$$

and to satisfy (3.23),

$$b_{i,j} = \sum_{i',j'\neq 0} a_{i,j;i+i',j+j'}. \quad (3.25)$$

To impose these constraints we use (3.13) to write (3.10) explicitly in the form given by (3.18) and obtain

$$b_{i,j} = (a^{(2)} - \delta^{(3)})v^*_{i+1/2,j} + (\beta^{(1)} - \gamma^{(4)})v^*_{i-1/2,j} + (\delta^{(2)} + \phi^{(2)})v^*_{i+1/2,j-1} + (\gamma^{(1)} - \phi^{(1)})v^*_{i-1/2,j-1} - (\alpha^{(3)} + \phi^{(3)})v^*_{i+1/2,j+1} - (\beta^{(4)} - \phi^{(4)})v^*_{i-1/2,j+1} - (\gamma^{(3)} - \epsilon^{(3)})u^*_{i+1,j+1/2} - (\delta^{(3)} - \gamma^{(4)})u^*_{i,j+1/2} + (\delta^{(4)} - \epsilon^{(4)})u^*_{i-1,j+1/2} - (\beta^{(2)} + \epsilon^{(2)})u^*_{i+1,j-1/2} - (\alpha^{(2)} - \beta^{(1)})u^*_{i,j-1/2} + (\alpha^{(1)} + \epsilon^{(1)})u^*_{i-1,j-1/2}, \quad (3.26)$$

$$\left.\begin{aligned}
a_{i,j;i+1,j} &= (\alpha^{(1)} - \delta^{(4)})v^*_{i+1/2,j} + (\delta^{(1)} + \phi^{(1)})v^*_{i+1/2,j-1} - (\alpha^{(4)} + \phi^{(4)})v^*_{i+1/2,j+1} \\
&\quad - (\gamma^{(4)} - \epsilon^{(4)})u^*_{i+1,j+1/2} - \delta^{(4)}u^*_{i,j+1/2} - (\beta^{(1)} + \epsilon^{(1)})u^*_{i+1,j-1/2} - \alpha^{(1)}u^*_{i,j-1/2}, \\
a_{i,j;i-1,j} &= (\beta^{(2)} - \gamma^{(3)})v^*_{i-1/2,j} + (\gamma^{(2)} - \phi^{(2)})v^*_{i-1/2,j-1} - (\beta^{(3)} - \phi^{(3)})v^*_{i-1/2,j+1} + \gamma^{(3)}u^*_{i,j+1/2} \\
&\quad + (\delta^{(3)} - \epsilon^{(3)})u^*_{i-1,j+1/2} + \beta^{(2)}u^*_{i,j-1/2} + (\alpha^{(2)} + \epsilon^{(2)})u^*_{i-1,j-1/2}, \\
a_{i,j;i,j+1} &= -\delta^{(2)}v^*_{i+1/2,j} - \gamma^{(1)}v^*_{i-1/2,j} - (\alpha^{(2)} + \phi^{(2)})v^*_{i+1/2,j+1} - (\beta^{(1)} - \phi^{(1)})v^*_{i-1/2,j+1} \\
&\quad - (\gamma^{(2)} - \epsilon^{(2)})u^*_{i+1,j+1/2} - (\delta^{(2)} - \gamma^{(1)})u^*_{i,j+1/2} + (\delta^{(1)} - \epsilon^{(1)})u^*_{i-1,j+1/2}, \\
a_{i,j;i,j-1} &= \alpha^{(3)}v^*_{i+1/2,j} + \beta^{(4)}v^*_{i-1/2,j} + (\delta^{(3)} + \phi^{(3)})v^*_{i+1/2,j-1} + (\gamma^{(4)} - \phi^{(4)})v^*_{i-1/2,j-1} \\
&\quad - (\beta^{(3)} + \epsilon^{(3)})u^*_{i+1,j-1/2} - (\alpha^{(3)} - \beta^{(4)})u^*_{i,j-1/2} + (\alpha^{(4)} + \epsilon^{(4)})u^*_{i-1,j-1/2}, \\
a_{i,j;i+1,j+1} &= -\delta^{(1)}v^*_{i+1/2,j} - (\alpha^{(1)} + \phi^{(1)})v^*_{i+1/2,j+1} - (\gamma^{(1)} - \epsilon^{(1)})u^*_{i+1,j+1/2} - \delta^{(1)}u^*_{i,j+1/2}, \\
a_{i,j;i-1,j-1} &= \beta^{(3)}v^*_{i-1/2,j} + (\gamma^{(3)} - \phi^{(3)})v^*_{i-1/2,j-1} + \beta^{(3)}u^*_{i,j-1/2} + (\alpha^{(3)} + \epsilon^{(3)})u^*_{i-1,j-1/2}, \\
a_{i,j;i-1,j+1} &= -\gamma^{(2)}v^*_{i-1/2,j} - (\beta^{(2)} - \phi^{(2)})v^*_{i-1/2,j+1} + \gamma^{(2)}u^*_{i,j+1/2} + (\delta^{(2)} - \epsilon^{(2)})u^*_{i-1,j+1/2}, \\
a_{i,j;i+1,j-1} &= \alpha^{(4)}v^*_{i+1/2,j} + (\delta^{(4)} + \phi^{(4)})v^*_{i+1/2,j-1} - (\beta^{(4)} + \epsilon^{(4)})u^*_{i+1,j-1/2} - \alpha^{(4)}u^*_{i,j-1/2}
\end{aligned}\right\} . \quad (3.27)$$

Applying first the constraint given by (3.25) for arbitrary $u^*$ and $v^*$ and simplifying, using (3.17), requires

$$\left.\begin{array}{l} \alpha^{(2)} - \delta^{(3)} = \beta^{(1)} - \gamma^{(4)} = 0 \\ \delta^{(3)} - \gamma^{(4)} = \alpha^{(2)} - \beta^{(1)} = 0 \\ \delta^{(2)} + \phi^{(2)} = \gamma^{(1)} - \phi^{(1)} = 1/8 \\ \alpha^{(3)} + \phi^{(3)} = \beta^{(4)} - \phi^{(4)} = 1/8 \\ \gamma^{(3)} - \epsilon^{(3)} = \delta^{(4)} - \epsilon^{(4)} = 1/8 \\ \beta^{(2)} + \epsilon^{(2)} = \alpha^{(1)} + \epsilon^{(1)} = 1/8 \end{array}\right\} . \tag{3.28}$$

To apply the constraint (3.24), each $a_{i,j;i-i',j-j'}$ is required to equal $-a_{i,j;i+i',j+j'}$ when all pairs of indices appearing in the former are incremented by $(i',j')$. This gives the following conditions:

$$\left.\begin{array}{l} \alpha^{(1)} - \delta^{(4)} + \beta^{(2)} - \gamma^{(3)} \\ \delta^{(1)} + \phi^{(1)} + \gamma^{(2)} - \phi^{(2)} \\ \alpha^{(4)} + \phi^{(4)} + \beta^{(3)} - \phi^{(3)} \\ \gamma^{(4)} - \epsilon^{(4)} - \gamma^{(3)} \\ \delta^{(4)} - \delta^{(3)} + \epsilon^{(3)} \\ \beta^{(1)} + \epsilon^{(1)} - \beta^{(2)} \\ \alpha^{(1)} - \alpha^{(2)} - \epsilon^{(2)} \end{array}\right\} = 0, \tag{3.29a}$$

$$\left.\begin{array}{l} \delta^{(2)} - \delta^{(3)} - \phi^{(3)} \\ \gamma^{(1)} - \gamma^{(4)} + \phi^{(4)} \\ \alpha^{(2)} + \phi^{(2)} - \alpha^{(3)} \\ \beta^{(1)} - \phi^{(1)} - \beta^{(4)} \\ \gamma^{(2)} - \epsilon^{(2)} + \beta^{(3)} + \epsilon^{(3)} \\ \delta^{(2)} - \gamma^{(1)} + \alpha^{(3)} - \beta^{(4)} \\ \delta^{(1)} - \epsilon^{(1)} + \alpha^{(4)} + \epsilon^{(4)} \end{array}\right\} = 0, \tag{3.29b}$$

$$\left.\begin{array}{l} \delta^{(1)} - \gamma^{(3)} + \phi^{(3)} \\ \alpha^{(1)} + \phi^{(1)} - \beta^{(3)} \\ \gamma^{(1)} - \epsilon^{(1)} - \beta^{(3)} \\ \delta^{(1)} - \alpha^{(3)} - \epsilon^{(3)} \end{array}\right\} = 0, \tag{3.29c}$$

$$\left.\begin{array}{l} \gamma^{(2)} - \delta^{(4)} - \phi^{(4)} \\ \beta^{(2)} - \phi^{(2)} - \alpha^{(4)} \\ \gamma^{(2)} - \beta^{(4)} - \epsilon^{(4)} \\ \delta^{(2)} - \epsilon^{(2)} - \alpha^{(4)} \end{array}\right\} = 0. \tag{3.29d}$$

Now (3.28) and (3.29) can be combined and all coefficients expressed in terms of $\epsilon^{(1)}$, $\epsilon^{(3)}$, $\phi^{(1)}$, and $\phi^{(3)}$:

$$\left.\begin{array}{llll} \alpha^{(1)} = 1/8 - \epsilon^{(1)}; & \alpha^{(2)} = 1/24; & \alpha^{(3)} = 1/8 - \phi^{(3)}; & \alpha^{(4)} = -1/24 + \epsilon^{(1)} + \phi^{(3)} \\ \beta^{(1)} = 1/24; & \beta^{(2)} = 1/24 + \epsilon^{(1)}; & \beta^{(3)} = 1/8 - \epsilon^{(1)} + \phi^{(1)}; & \beta^{(4)} = 1/24 - \phi^{(1)} \\ \gamma^{(1)} = 1/8 + \phi^{(1)}; & \gamma^{(2)} = -1/24 - \epsilon^{(3)} - \phi^{(1)}; & \gamma^{(3)} = 1/8 + \epsilon^{(3)}; & \gamma^{(4)} = 1/24 \\ \delta^{(1)} = 1/8 + \epsilon^{(3)} - \phi^{(3)}; & \delta^{(2)} = 1/24 + \phi^{(3)}; & \delta^{(3)} = 1/24; & \delta^{(4)} = 1/24 - \epsilon^{(3)} \\ \epsilon^{(2)} = 1/12 - \epsilon^{(1)}; & \epsilon^{(4)} = -1/12 - \epsilon^{(3)} & & \\ \phi^{(2)} = 1/12 - \phi^{(3)}; & \phi^{(4)} = -1/12 - \phi^{(1)} & & \end{array}\right\} . \tag{3.30}$$

In the square grid case we should require that $u^*$ and $v^*$ be treated in an identical manner under rotation of the axes and thus, for example, that the coefficient of $q_{i,j}u^*_{i,j+1/2}$ in the $v$ equation at $(i + 1/2, j)$ be equal to that of the product $q_{i,j}v^*_{i+1/2,j}$ in the $u$ equation at $(i, j - 1/2)$. Applying this requirement to all $qu^*$ and $qv^*$ terms leads to the following constraints:

$$\left.\begin{array}{llll} \alpha^{(1)} = \beta^{(4)}; & \alpha^{(2)} = \beta^{(1)}; & \alpha^{(3)} = \beta^{(2)}; & \alpha^{(4)} = \beta^{(3)} \\ \delta^{(1)} = \alpha^{(4)}; & \delta^{(2)} = \alpha^{(1)}; & \delta^{(3)} = \alpha^{(2)}; & \delta^{(4)} = \alpha^{(3)} \end{array}\right\} . \tag{3.31}$$

From (3.30) and (3.31), we obtain

$$\epsilon^{(3)} = -\epsilon^{(1)}; \quad \phi^{(3)} = 1/12 - \epsilon^{(1)}; \quad \phi^{(1)} = \epsilon^{(1)} - 1/12. \tag{3.32}$$

If we now choose, simply for reasons of symmetry between $\epsilon$ and $\phi$,

$$\epsilon^{(1)} = 1/24, \tag{3.33}$$

the terms in (3.13) are completely specified, as

$$\left.\begin{array}{rl} \epsilon_{i+1/2,j+1/2} = & 1/24[q_{i+1,j+1} + q_{i,j+1} - q_{i,j} - q_{i+1,j}] \\ \phi_{i+1/2,j+1/2} = & 1/24[-q_{i+1,j+1} + q_{i,j+1} + q_{i,j} - q_{i+1,j}] \\ \alpha_{i,j+1/2} = & 1/24[2q_{i+1,j+1} + q_{i,j+1} + 2q_{i,j} + q_{i+1,j}] \\ \beta_{i,j+1/2} = & 1/24[q_{i,j+1} + 2q_{i-1,j+1} + q_{i-1,j} + 2q_{i,j}] \\ \gamma_{i,j+1/2} = & 1/24[2q_{i,j+1} + q_{i-1,j+1} + 2q_{i-1,j} + q_{i,j}] \\ \delta_{i,j+1/2} = & 1/24[q_{i+1,j+1} + 2q_{i,j+1} + q_{i,j} + 2q_{i+1,j}] \end{array}\right\} . \tag{3.34}$$

The rate of increase in the total kinetic energy due to the pressure gradient force is seen from (3.7) to be

$$-\sum_{h\ \text{pts}} (\Phi\nabla\cdot\mathbf{v}^*)_{i+1/2,j+1/2}. \tag{3.35}$$

The rate of increase of total potential energy at the point $(i + ½, j + ½)$ is obtained by multiplying (3.1) by $\Phi_{i+1/2,j+1/2}$. When we choose

$$h^{(u)}_{i,j+1/2} = (\bar{h}^i)_{i,j+1/2}, \tag{3.36}$$

$$h^{(v)}_{i+1/2,j} = (\bar{h}^j)_{i+1/2,j}, \tag{3.37}$$

in the definition of $u^*$ and $v^*$ given by (3.3) and (3.4), where overbars $-i$ and $-j$ denote the arithmetic average of two neighboring points in $x$ and $y$ directions respectively, we obtain

$$\sum_{h\ \text{pts}} \frac{\partial}{\partial t}[½gh^2 + ghh_s]_{i+1/2,j+1/2} + \sum_{h\ \text{pts}} (\Phi\nabla\cdot\mathbf{v}^*)_{i+1/2,j+1/2} = 0. \tag{3.38}$$

From (3.35) and (3.38), it is clear that the finite-difference form of the pressure gradient force in (3.5) and (3.6) does not cause any production of total energy.

With $h^{(u)}$ and $h^{(v)}$ given by (3.36) and (3.37), the form for $K$ is now fixed by the requirement for conservation of total kinetic energy given by (3.9). Making use of the continuity equation (3.1) and the fact that for any variables $a$, $b$ on a staggered grid

$$\sum_{a\ \text{pts}} a_{i,j}(\bar{b}^i)_{i,j} = \sum_{b\ \text{pts}} b_{i+1/2,j}(\bar{a}^i)_{i+1/2,j}, \tag{3.39}$$

Eq. (3.9) can be written

$$\sum_{h\ \text{pts}} \left[\left(\frac{\partial h}{\partial t}\right)_{i+1/2,j+1/2} \times (\overline{½u^2}^i + \overline{½v^2}^j - K)_{i+1/2,j+1/2}\right] = 0, \tag{3.40}$$

and thus

$$K_{i+1/2,j+1/2} = (\overline{½u^2}^i + \overline{½v^2}^j)_{i+1/2,j+1/2}. \tag{3.41}$$

In summary, use of (3.11), (3.12), (3.15), (3.34), (3.36), (3.37) and (3.41) in (3.1)–(3.6) gives a potential enstrophy and energy conserving scheme for the shallow water equations.

## 4. Potential enstrophy conservation schemes for the case of no mass flux divergence

In the previous section, a space finite-difference scheme was derived that conserved energy and potential enstrophy for the general case of divergent mass flux. In order to show the relationship between that scheme and certain other recently proposed schemes, we consider in this section schemes that have the same energy conservation properties but conserve potential enstrophy only in the special case of nondivergent mass flux, for which $\nabla\cdot\mathbf{v}^* = 0$. In such flow we can define a streamfunction $\psi^*$ for the mass flux such that

$$\mathbf{v}^* = \mathbf{k}\times\nabla\psi^*. \tag{4.1}$$

Eq. (2.9) can then be expressed as

$$\frac{\partial}{\partial t}(hq) + J(\psi^*,q) = 0, \tag{4.2}$$

where the Jacobian is defined, in the customary way, as

$$J(a,b) \equiv \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}.$$

It is now well known (Arakawa, 1966, 1970) that for horizontal nondivergent flow the use of the finite-difference Arakawa-Jacobian denoted by $J_A$ to represent the advection term $J(\psi,\zeta)$ in the vorticity equation maintains the conservation of enstrophy and kinetic energy. This result can be applied directly to the Jacobian in (4.2) to give a finite-difference representation for the vorticity equation that guarantees conservation of potential enstrophy for flow in which $\nabla\cdot\mathbf{v}^* = 0$. A scheme for the momentum equation (2.1) then conserves potential enstrophy if the finite-difference potential vorticity equation derived from it reduces to (4.2) with $J_A(\psi^*,q)$ when $\nabla\cdot\mathbf{v}^* = 0$.

The Arakawa Jacobian is presented below for convenience:

$$\begin{aligned}
[J_A(q,\psi^*)]_{i,j} = -\frac{1}{12d^2}[&(\psi^*_{i,j-1} + \psi^*_{i+1,j-1} - \psi^*_{i,j+1} - \psi^*_{i+1,j+1})(q_{i+1,j} + q_{i,j}) - (\psi^*_{i-1,j-1} + \psi^*_{i,j-1} - \psi^*_{i-1,j+1} - \psi^*_{i,j+1}) \\
&\times (q_{i,j} + q_{i-1,j}) + (\psi^*_{i+1,j} + \psi^*_{i+1,j+1} - \psi^*_{i-1,j} - \psi^*_{i-1,j+1})(q_{i,j+1} + q_{i,j}) - (\psi^*_{i+1,j-1} + \psi^*_{i+1,j} \\
&- \psi^*_{i-1,j-1} - \psi^*_{i-1,j})(q_{i,j} + q_{i,j-1}) + (\psi^*_{i+1,j} - \psi^*_{i,j+1})(q_{i+1,j+1} + q_{i,j}) - (\psi^*_{i,j-1} - \psi^*_{i-1,j})(q_{i,j} + q_{i-1,j-1}) \\
&+ (\psi^*_{i,j+1} - \psi^*_{i-1,j})(q_{i-1,j+1} + q_{i,j}) - (\psi^*_{i+1,j} - \psi^*_{i,j-1})(q_{i,j} + q_{i+1,j-1})].
\end{aligned} \tag{4.3}$$

With

$$u^*_{i,j+1/2} = d^{-1}(\psi^*_{i,j} - \psi^*_{i,j+1}) \tag{4.4}$$

and

$$v^*_{i+1/2,j} = d^{-1}(\psi^*_{i+1,j} - \psi^*_{i,j}), \tag{4.5}$$

the right-hand side of (3.10) can be rewritten as

$$
\begin{aligned}
-d^{-2}[&\psi^*_{i+1,j}(-\alpha_{i,j-1/2} + \delta_{i,j+1/2} + \gamma_{i+1,j+1/2} - \beta_{i+1,j-1/2} - \epsilon_{i+1/2,j-1/2} - \epsilon_{i+1/2,j+1/2}) \\
&+ \psi^*_{i-1,j}(\beta_{i,j-1/2} - \gamma_{i,j+1/2} - \delta_{i-1,j+1/2} + \alpha_{i-1,j-1/2} + \epsilon_{i-1/2,j-1/2} + \epsilon_{i-1/2,j+1/2}) \\
&+ \psi^*_{i,j-1}(-\alpha_{i,j+1/2} + \beta_{i,j+1/2} - \delta_{i,j+1/2} + \gamma_{i,j+1/2} - \phi_{i+1/2,j+1/2} - \phi_{i-1/2,j+1/2}) \\
&+ \psi^*_{i,j-1}(-\gamma_{i,j-1/2} + \delta_{i,j-1/2} + \alpha_{i,j-1/2} - \beta_{i,j-1/2} + \phi_{i+1/2,j-1/2} + \phi_{i-1/2,j-1/2}) \\
&+ \psi^*_{i+1,j+1}(\alpha_{i,j+1/2} - \gamma_{i+1,j+1/2} + \epsilon_{i+1/2,j+1/2} + \phi_{i+1/2,j+1/2}) \\
&+ \psi^*_{i-1,j+1}(-\beta_{i,j+1/2} + \delta_{i-1,j+1/2} - \epsilon_{i-1/2,j+1/2} + \phi_{i-1/2,j+1/2}) \\
&+ \psi^*_{i-1,j-1}(\gamma_{i,j-1/2} - \alpha_{i-1,j-1/2} - \epsilon_{i-1/2,j-1/2} - \phi_{i-1/2,j-1/2}) \\
&+ \psi^*_{i+1,j-1}(-\delta_{i,j-1/2} + \beta_{i+1,j-1/2} + \epsilon_{i+1/2,j-1/2} - \phi_{i+1/2,j-1/2})].
\end{aligned} \tag{4.6}
$$

Equating (4.6) with the right-hand side of (4.3) gives four independent constraints on $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$, $\phi$, as functions of $q$:

$$
\begin{aligned}
-\alpha_{i,j-1/2} + \delta_{i,j+1/2} + \gamma_{i+1,j+1/2} \\
- \beta_{i+1,j-1/2} - \epsilon_{i+1/2,j-1/2} - \epsilon_{i+1/2,j+1/2} \\
= -\tfrac{1}{12}(q_{i+1,j-1} + q_{i,j-1} - q_{i+1,j+1} - q_{i,j+1}),
\end{aligned} \tag{4.7a}
$$

$$
\begin{aligned}
-\alpha_{i,j+1/2} + \beta_{i,j+1/2} - \delta_{i,j+1/2} \\
+ \gamma_{i,j+1/2} - \phi_{i+1/2,j+1/2} - \phi_{i-1/2,j+1/2} \\
= -\tfrac{1}{12}(q_{i+1,j+1} + q_{i+1,j} - q_{i-1,j+1} - q_{i-1,j}),
\end{aligned} \tag{4.7b}
$$

$$
\alpha_{i,j+1/2} - \gamma_{i+1,j+1/2} + \epsilon_{i+1/2,j+1/2} + \phi_{i+1/2,j+1/2} = -\tfrac{1}{12}(q_{i+1,j} - q_{i,j+1}), \tag{4.7c}
$$

$$
-\beta_{i,j+1/2} + \delta_{i-1,j+1/2} - \epsilon_{i-1/2,j+1/2} + \phi_{i-1/2,j+1/2} = -\tfrac{1}{12}(q_{i,j+1} - q_{i-1,j}). \tag{4.7d}
$$

If the four equations (4.7) are required to hold for every $(i,j)$, we can solve for $\alpha$, $\beta$, $\gamma$ and $\delta$ in terms of $q$, both explicitly and implicitly through $\epsilon$ and $\phi$. To this end, we let

$$
\tfrac{1}{2}(\alpha_{i,j+1/2} + \gamma_{i+1,j+1/2}) \equiv A_{i+1/2,j+1/2}. \tag{4.8}
$$

Then, using (4.7c),

$$
\begin{aligned}
\alpha_{i,j+1/2} &= A_{i+1/2,j+1/2} + \tfrac{1}{2}(\alpha_{i,j+1/2} - \gamma_{i+1,j+1/2}) \\
&= A_{i+1/2,j+1/2} - \tfrac{1}{24}(q_{i+1,j} - q_{i,j+1}) - \tfrac{1}{2}(\epsilon + \phi)_{i+1/2,j+1/2},
\end{aligned} \tag{4.9}
$$

$$
\gamma_{i+1,j+1/2} = A_{i+1/2,j+1/2} + \tfrac{1}{24}(q_{i+1,j} - q_{i,j+1}) + \tfrac{1}{2}(\epsilon + \phi)_{i+1/2,j+1/2}. \tag{4.10}
$$

Similarly, we let

$$
\tfrac{1}{2}(\beta_{i,j+1/2} + \delta_{i-1,j+1/2}) \equiv B_{i-1/2,j+1/2}. \tag{4.11}
$$

Then, using (4.7d),

$$
\begin{aligned}
\beta_{i,j+1/2} &= B_{i-1/2,j+1/2} + \tfrac{1}{2}(\beta_{i,j+1/2} - \delta_{i-1,j+1/2}) \\
&= B_{i-1/2,j+1/2} + \tfrac{1}{24}(q_{i,j+1} - q_{i-1,j}) - \tfrac{1}{2}(\epsilon - \phi)_{i-1/2,j+1/2},
\end{aligned} \tag{4.12}
$$

$$
\delta_{i-1,j+1/2} = B_{i-1/2,j+1/2} - \tfrac{1}{24}(q_{i,j+1} - q_{i-1,j}) + \tfrac{1}{2}(\epsilon - \phi)_{i-1/2,j+1/2}. \tag{4.13}
$$

As (4.9), (4.10), (4.12) and (4.13) must hold for every $(i,j)$, they can be used in (4.7a) to give

$$
\begin{aligned}
(A + B)_{i+1/2,j+1/2} - \tfrac{1}{8}[q_{i+1,j+1} + q_{i+1,j} + q_{i,j+1} + q_{i,j}] \\
= (A + B)_{i+1/2,j-1/2} - \tfrac{1}{8}[q_{i+1,j} + q_{i+1,j-1} + q_{i,j} + q_{i,j-1}].
\end{aligned} \tag{4.14}
$$

By the same method (4.9), (4.10), (4.12) and (4.13) can be used in (4.7b) to give

$$
\begin{aligned}
(A + B)_{i+1/2,j+1/2} - \tfrac{1}{8}[q_{i+1,j+1} + q_{i,j+1} + q_{i+1,j} + q_{i,j}] \\
= (A + B)_{i-1/2,j+1/2} - \tfrac{1}{8}[q_{i,j+1} + q_{i-1,j+1} + q_{i,j} + q_{i-1,j}].
\end{aligned} \tag{4.15}
$$

The same expression is thus shown to have a constant value under variation of the $i$ index by (4.14) and of the $j$ index by (4.15). The constant can be taken equal to zero without loss of generality and we have, for all $i$ and $j$

$$
(A + B)_{i+1/2,j+1/2} = \tfrac{1}{8}(q_{i+1,j+1} + q_{i,j+1} + q_{i+1,j} + q_{i,j}). \tag{4.16}
$$

Instead of solving (4.16) for either $A$ or $B$, symmetry will be preserved by writing each in terms of a new variable. For this purpose we define

$$
C_{i+1/2,j+1/2} \equiv \tfrac{1}{2}(A - B)_{i+1/2,j+1/2}. \tag{4.17}
$$

Then combining (4.16) and (4.17), we obtain

$$
A_{i+1/2,j+1/2} = C_{i+1/2,j+1/2} + \tfrac{1}{16}(q_{i+1,j+1} + q_{i,j+1} + q_{i+1,j} + q_{i,j}). \tag{4.18}
$$

$$
B_{i+1/2,j+1/2} = -C_{i+1/2,j+1/2} + \tfrac{1}{16}(q_{i+1,j+1} + q_{i,j+1} + q_{i+1,j} + q_{i,j}). \tag{4.19}
$$

These expressions, used in (4.9), (4.10), (4.12) and (4.13), allow us to write $\alpha$, $\beta$, $\gamma$, $\delta$, respectively, in terms of the surrounding $q$, the as-yet-undetermined functions $\epsilon$ and $\phi$, and the new variable $C$. The final form is given below:

$$\alpha_{i,j+1/2} = C_{i+1/2,j+1/2} + \tfrac{1}{48}[5q_{i,j+1} + 3(q_{i,j} + q_{i+1,j+1}) + q_{i+1,j}] - \tfrac{1}{2}(\epsilon_{i+1/2,j+1/2} + \phi_{i+1/2,j+1/2}),$$

$$\beta_{i,j+1/2} = -C_{i-1/2,j+1/2} + \tfrac{1}{48}[5q_{i,j+1} + 3(q_{i,j} + q_{i-1,j+1}) + q_{i-1,j}] - \tfrac{1}{2}(\epsilon_{i-1/2,j+1/2} - \phi_{i-1/2,j+1/2}),$$

$$\gamma_{i,j+1/2} = C_{i-1/2,j+1/2} + \tfrac{1}{48}[5q_{i,j} + 3(q_{i,j+1} + q_{i-1,j}) + q_{i-1,j+1}] + \tfrac{1}{2}(\epsilon_{i-1/2,j+1/2} + \phi_{i-1/2,j+1/2}),$$

$$\delta_{i,j+1/2} = -C_{i+1/2,j+1/2} + \tfrac{1}{48}[5q_{i,j} + 3(q_{i,j+1} + q_{i+1,j}) + q_{i+1,j+1}] + \tfrac{1}{2}(\epsilon_{i+1/2,j+1/2} - \phi_{i+1/2,j+1/2}). \quad (4.20)$$

There is thus a family of schemes of the form (3.5), (3.6), with $\alpha$, $\beta$, $\gamma$ and $\delta$ given by (4.20) and $K, h^{(u)}, h^{(v)}$ related by (3.9). All schemes in the family guarantee conservation of kinetic energy in the $\nabla\cdot\mathbf{v}^* \neq 0$ case, and of potential enstrophy in the $\nabla\cdot\mathbf{v}^* = 0$ case.

It can be mentioned here that the scheme for the differencing of the momentum advection terms derived by Sadourny and subsequently tested by the European Centre for Medium Range Weather Forecasts (Burridge and Haseler, 1977) is a member of this family. Applying their scheme to a square grid and using the present indexing and notation, we have

$$\left.\begin{aligned} \alpha_{i,j+1/2} &= \tfrac{1}{12}(q_{i,j+1} + q_{i,j} + q_{i+1,j+1}) \\ \beta_{i,j+1/2} &= \tfrac{1}{12}(q_{i,j+1} + q_{i,j} + q_{i-1,j+1}) \\ \gamma_{i,j+1/2} &= \tfrac{1}{12}(q_{i,j+1} + q_{i,j} + q_{i-1,j}) \\ \delta_{i,j+1/2} &= \tfrac{1}{12}(q_{i,j+1} + q_{i,j} + q_{i+1,j}) \\ \epsilon_{i+1/2,j+1/2} &= \phi_{i+1/2,j+1/2} = 0 \end{aligned}\right\} . \quad (4.21)$$

By comparison with (4.20), these definitions correspond to a choice of

$$C_{i+1/2,j+1/2} = \tfrac{1}{48}[q_{i,j+1} - q_{i,j} + q_{i+1,j} - q_{i+1,j+1}]. \quad (4.22)$$

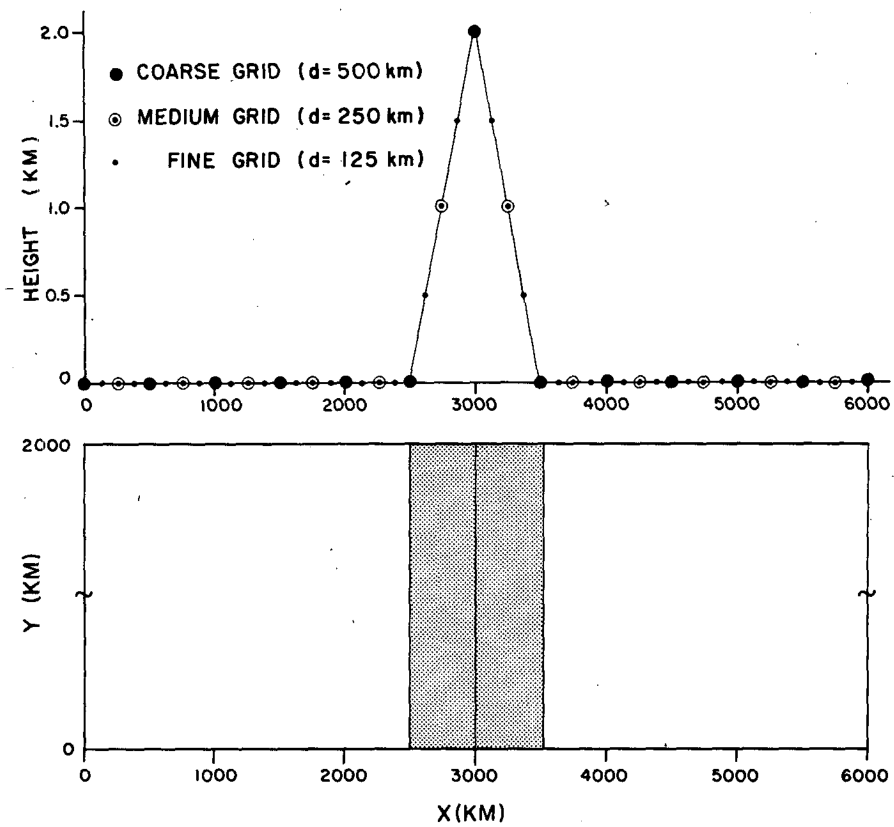


FIG. 2. The plain view of the topography and its recognition by the course, medium and fine grids.

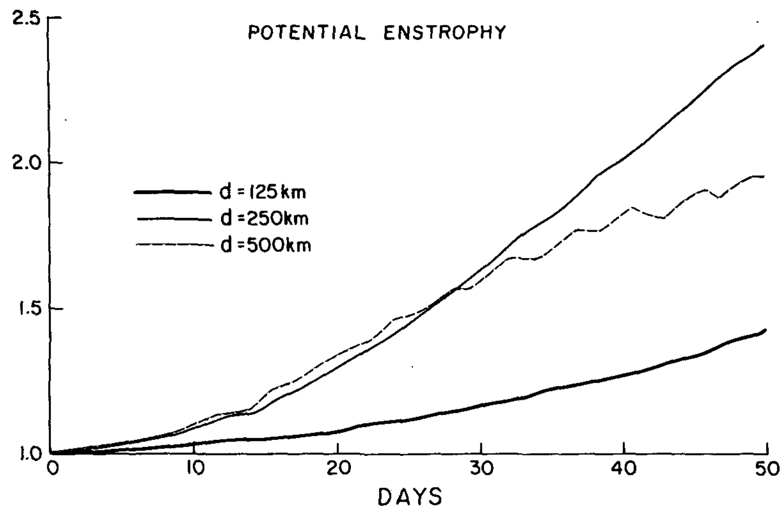


FIG. 3. Examples of the potential enstrophy increase in time using the PE non-conserving scheme for different grid sizes.

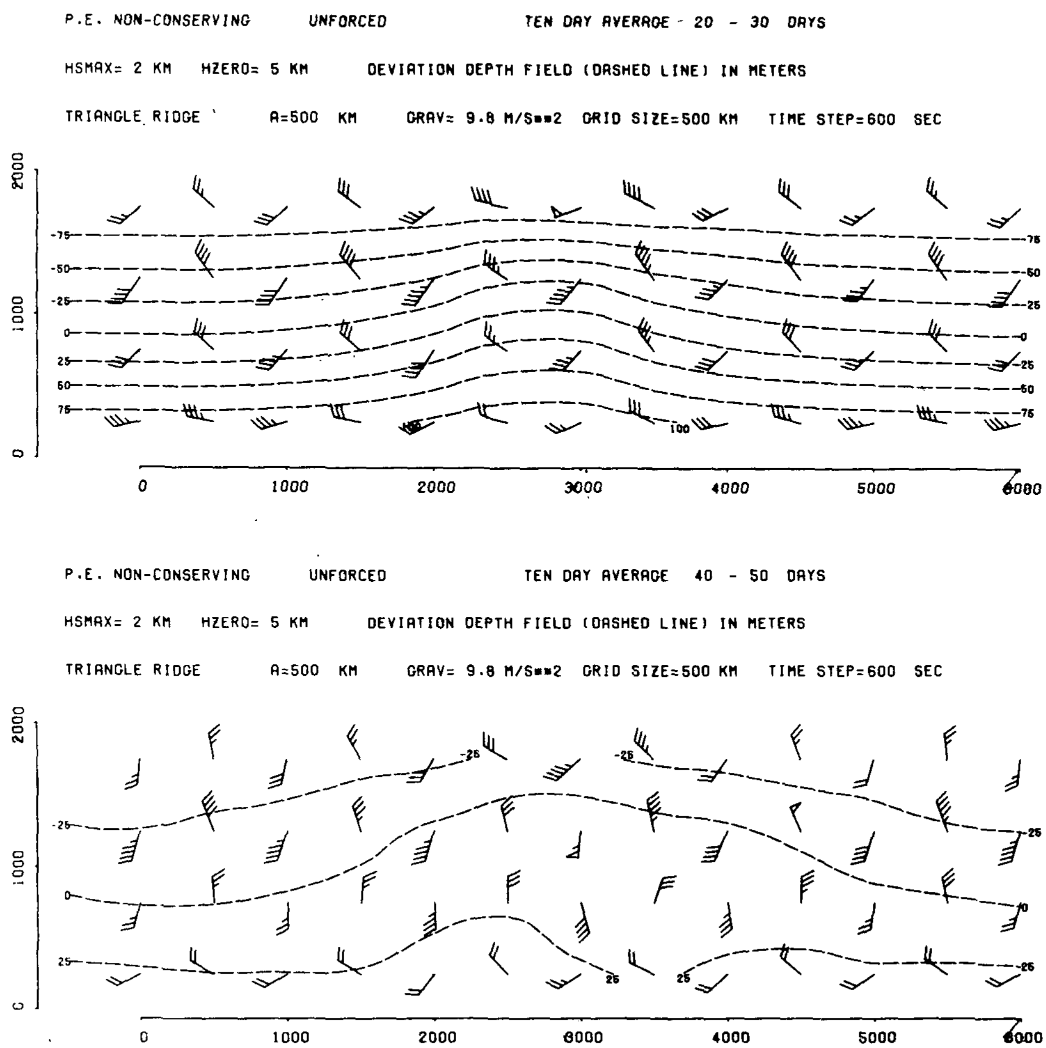


FIG. 4. Result of the integration with the coarse grid using the PE non-conserving scheme averaged over the 10-day periods from day 20 to 30 (the upper figure) and day 40 to 50 (the lower figure).

As (4.21) is not equivalent to the scheme derived in Section 3, given by (3.34), it is clear that the choice of $C$ given by (4.22) does not utilize the freedom remaining in (4.20) to achieve conservation of potential enstrophy for a general flow.

## 5. Numerical tests of the potential enstrophy conserving scheme

In this section the properties of the scheme derived in Section 3 (hereafter referred to as the PE conserving scheme) are compared with those of a scheme that conserves (potential) enstrophy only for purely horizontal nondivergent flow (hereafter referred to as the PE non-conserving scheme), by means of numerical integrations. The PE non-conserving scheme corresponds to that previously used in the UCLA general circulation model, but applied to the shallow water equations (Arakawa and Lamb, 1977, Section III C).

The domain used in the numerical experiments is bounded by $y = 0$ and $y = 2000$ km, where rigid walls are assumed, and by $x = 0$ and $x = 6000$ km, where cyclic boundary conditions are applied. In the first set of experiments, the mean height of the free surface $H_0$ is 5 km; the acceleration of gravity $g$ is 9.8 m $s^{-2}$; and the Coriolis parameter $f$ is $10^{-4}$ $s^{-1}$. The bottom topography is a narrow ridge, centered at $x = 3000$ km, that uniformly extends across the channel in $y$ and has a triangular shape in $x$, with a maximum height of 2 km and a bottom width of 1000 km. Experiments were performed with three different grid sizes: $d = 500$, 250 and 125 km. Fig. 2 shows the plan view of the topography and how it is recognized by the $h$-points of each of the three grids. At the lateral boundaries, the rigid wall con-

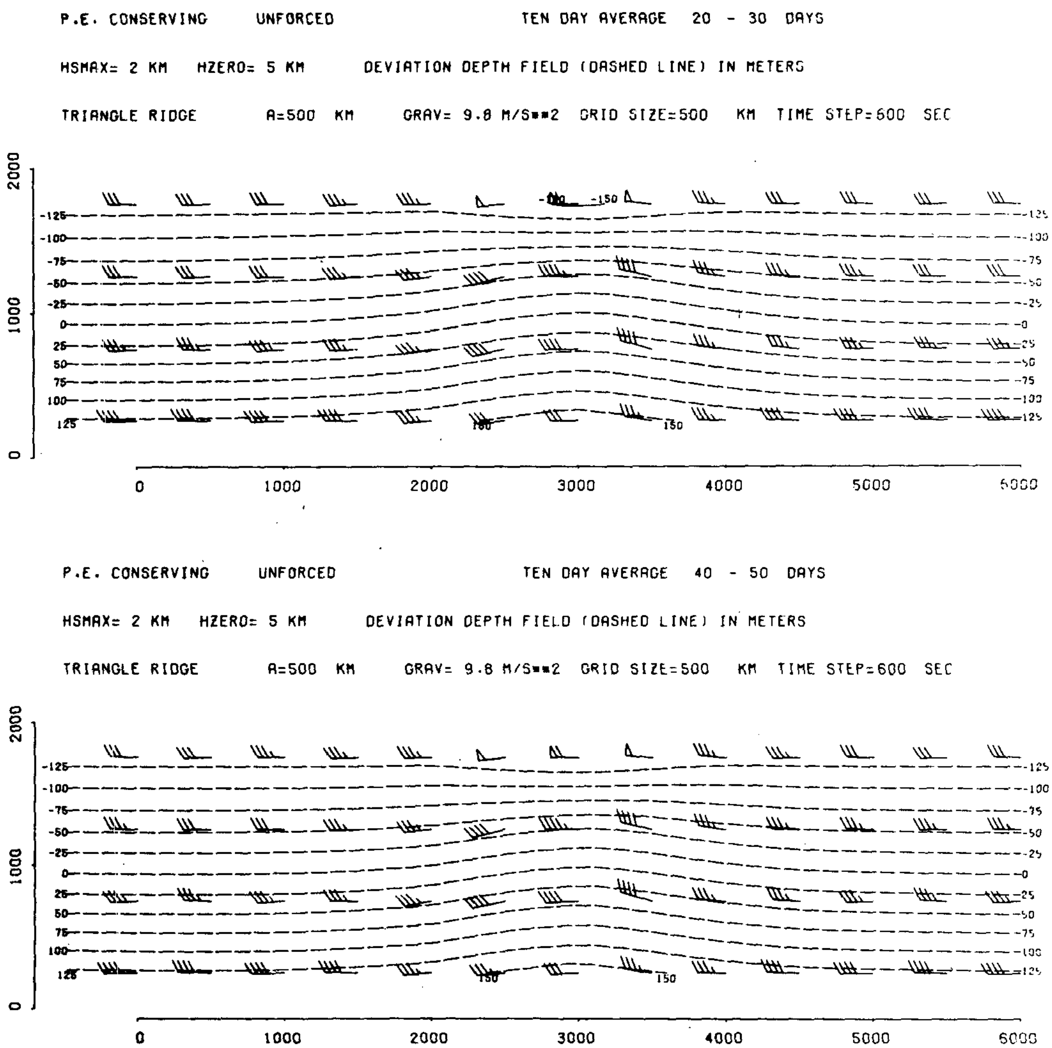


FIG. 5. As in Fig. 4 except using the PE conserving scheme.

dition $v = 0$ and a computational boundary condition $\zeta = 0$ are applied. The second-order Heun scheme is used for the initial time step and once in every 300 leapfrog time steps. The time interval $\Delta t$ is 10 min for $d = 500$ km, 5 min for $d = 250$ km and 25 min for $d = 125$ km. The initial conditions are a uniform zonal current of 20 m $s^{-1}$ and a horizontal free surface.

Fig. 3 shows the potential enstrophy increase in time with the PE non-conserving scheme. From this figure it is clear that no improvement in conservation of potential enstrophy is achieved by decreasing the grid size from 500 to 250 km. Even with $d = 125$ km there is a considerable increase of potential enstrophy in time.

Fig. 4 shows the wind vector based on $\bar{u}^i$ and $\bar{v}^j$ at $h$ points and the differential height of the free surface, $h + h_s - H_0$, for the case with $d = 500$ km, averaged over the 10-day periods from day 20 to 30 and from day 40 to 50. The wind vector field shows that a large amount of meridional kinetic energy exists in the smallest resolvable scale in $x$, even for 10-day average, and continues to grow in time. Total energy, on the other hand, is practically conserved throughout the integration and the growth of meridional kinetic energy takes place at the ex-

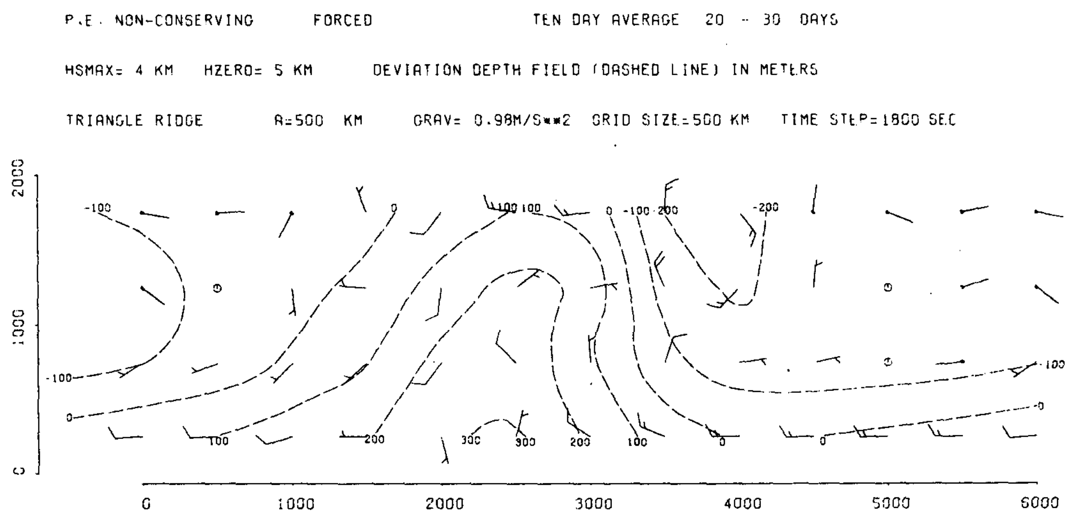


FIG. 6a. Results of the integration with forcing and friction averaged over the 10-day period from day 20 to 30 using the PE nonconserving scheme with the course grid.

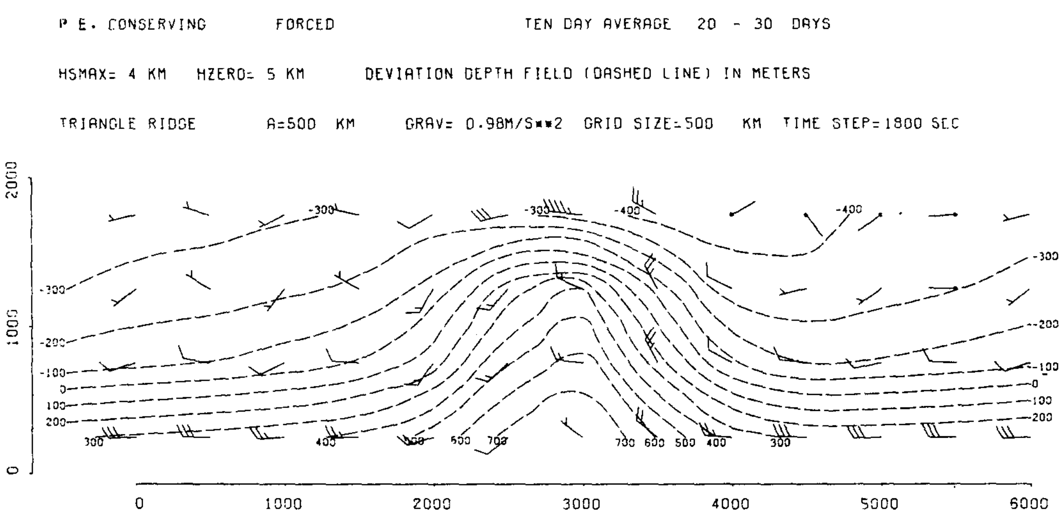


FIG. 6b. Results of the integration with forcing and friction averaged over the 10-day period from day 20 to 30 using the PE conserving scheme with the coarse grid.

pense of zonal kinetic energy and, to a lesser extent, of available potential energy.

Fig. 5 corresponds to Fig. 4 but with the new PE conserving scheme. The computational noise in the wind field is now drastically reduced.

To compare solutions of the two schemes in the simulation of a statistically steady state, we performed a second set of experiments. A surface stress linearly proportional to the wind, with coefficient $0.25 \times 10^{-5}$ s$^{-1}$, and a uniform westerly momentum generation of $2.5 \times 10^{-5}$ m s$^{-2}$ per unit mass are introduced. The maximum height of the mountain is increased to 4 km, and $g$ is reduced to 0.98 m s$^{-2}$ to partially include the effect of stratification. All other parameters, domain geometry and numerical procedures are the same as in the previous experiments, except that $\Delta t$ is increased by the factor of 3 for each grid size and the Euler scheme with $2\Delta t$ time interval is used for the friction term. The initial condition has no motion and horizontal free surface $H = H_0$. Figs. 6–8 show the time averages for the period from day 20 to 30.

Figs. 6a and 6b show that with the coarse grid, $d$ = 500 km, the PE non-conserving scheme produces a weak, relatively disorganized flow, while the PE conserving scheme produces an organized, dominantly westerly flow with a continuous meandering jet stream. The PE non-conserving scheme produces a weak ridge at the west side of the mountain, while the new scheme produces a stronger ridge almost right over the mountain. From Figs. 7a and 7b it is seen that even with the medium grid, $d$ = 250 km, the situation does not change significantly. Figs. 8a and 8b, however, show that with

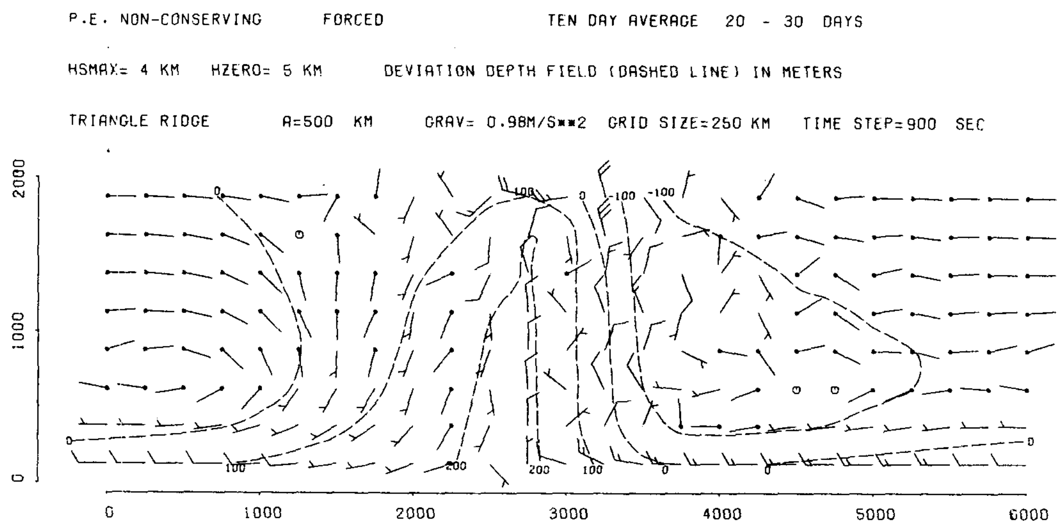


FIG. 7a. As in Fig. 6a except with the medium grid.

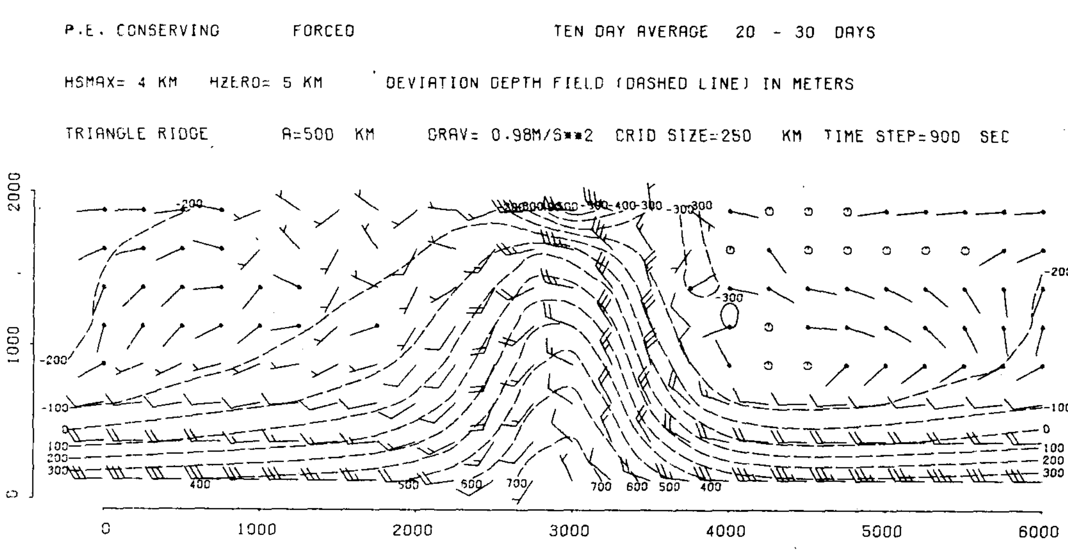


FIG. 7b. As in Fig. 6b except with the medium grid.

the fine grid, $d = 125$ km, the two schemes produce an almost identical field. By comparing Figs. 6a and 7a with Fig. 8a and Figs. 6b and 7b with Fig. 8b, we see that, as the grid size is reduced, the characteristics of the produced field change less with the PE conserving scheme than with the PE non-conserving scheme. This indicates that the solution with the PE conserving scheme is in an approximately convergent range even with the coarse grid, while that with the PE non-conserving scheme is not.

## 6. Summary and further comments

A second-order space difference scheme for the shallow water equations that conserves both potential enstrophy and total energy under the existence of bottom topography has been derived and tested. Comparison by means of numerical experiment with a scheme that conserves (potential) enstrophy only for purely horizontal nondivergent flow demonstrated the considerable superiority of the newly designed potential enstrophy conserving scheme, not only in suppressing a spurious energy cascade but also in determining the overall flow regime.

In addition, a family of schemes has been derived that conserves total energy for a general flow but potential enstrophy only for flow in which there is no mass flux divergence. It is pointed out that the newly designed scheme is a unique member of this family, that has the constraint of potential enstrophy conservation for flow with mass flux divergence.

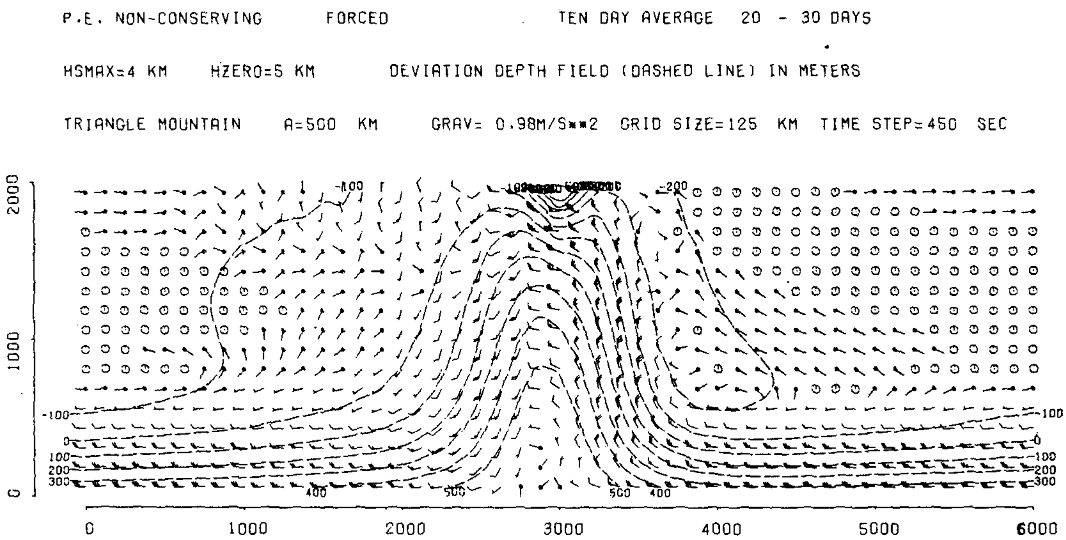


FIG. 8a. As in Fig. 6a except with the fine grid.

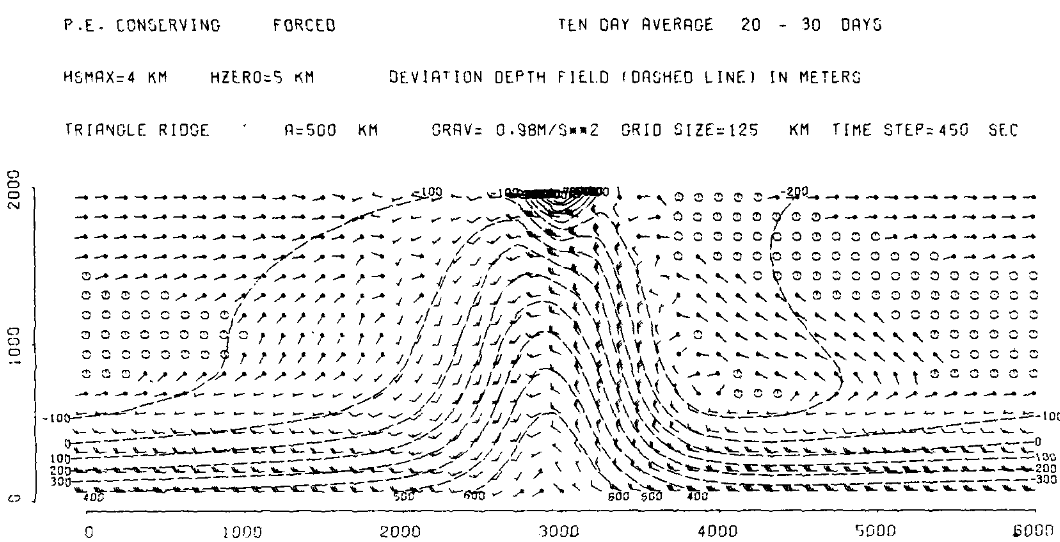


FIG. 8b. As in Fig. 6b except with the fine grid.

In the newly designed scheme, momentum conservation under advective process (i.e., without bottom topography and friction) is not formally guaranteed when the flow is divergent. Further numerical experiments under such conditions have demonstrated, however, that momentum was conserved by the scheme with sufficient accuracy.

The analogous second-order scheme for a spherical grid is presented in the Appendix. The fourth-order version of the scheme, both for square and spherical grids, has been derived by K. Takano of UCLA.

When the scheme presented here is applied to a three-dimensional model with a non-material surface vertical coordinate, linear computational instability of a meridionally propagating inertia-gravity wave can occur. Existence of such instability was first pointed out by Hollingsworth and Kållberg (personal communication) of European Centre for Medium Range Forecasts. We have found that use of

$$K_{i+1/2,j+1/2} = \overline{[\tfrac{1}{4}u_{j+1/2}^2 + \tfrac{1}{16}(u_{j+3/2} + u_{j-1/2})^2]}^i_{i+1/2} + \overline{[\tfrac{1}{4}v_{i+1/2}^2 + \tfrac{1}{16}(v_{i+3/2} + v_{i-1/2})^2]}^j_{j+1/2} \quad (6.1)$$

instead of (3.41) practically eliminates the instability. More details on this matter will be presented in a forthcoming paper.

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## APPENDIX

## The Potential Enstrophy Conserving Scheme for the Shallow Water Equations on a Spherical Grid

### 1. The governing equations in orthogonal curvilinear coordinates

Let the orthogonal curvilinear coordinates be $\xi$ and $\eta$. Let the actual distances corresponding to $d\xi$ and $d\eta$ be $(ds)_\xi$ and $(ds)_\eta$, respectively, and define the metric factors, $m$, $n$ such that

$$\left.\begin{aligned}(ds)_\xi &= (1/m)d\xi\\ (ds)_\eta &= (1/n)d\eta\end{aligned}\right\}. \quad (A1)$$

Let the component of $\mathbf{v}$ in $\xi$ be $u$ and the component of $\mathbf{v}$ in $\eta$ be $v$. In the spherical coordinates, $\xi = \lambda$ (longitude), and $\eta = \varphi$ (latitude), $1/m = a\cos\varphi$ and $1/n = a$, where $a$ is the radius of the earth.

The shallow water equations (2.1) and (2.2) on an orthogonal curvilinear grid take the form

$$\frac{\partial}{\partial t}\left(\frac{u}{m}\right) - q\frac{hv}{m} + \frac{\partial}{\partial \xi}(K + \Phi) = 0, \quad (A2)$$

$$\frac{\partial}{\partial t}\left(\frac{v}{n}\right) + q\frac{hu}{n} + \frac{\partial}{\partial \eta}(K + \Phi) = 0, \quad (A3)$$

$$\frac{\partial}{\partial t}\left(\frac{h}{nm}\right) + \frac{\partial}{\partial \xi}\left(h\frac{u}{n}\right) + \frac{\partial}{\partial \eta}\left(h\frac{v}{m}\right) = 0, \quad (A4)$$

where $q$, $K$ and $\Phi$ are defined in Section 2. The vorticity $\zeta = \mathbf{k}\cdot\nabla\times\mathbf{v}$ can be expressed as

$$mn\left[\frac{\partial}{\partial \xi}\frac{v}{n} - \frac{\partial}{\partial \eta}\frac{u}{m}\right]. \quad (A5)$$

Multiplying (A2) by $hu/n$, (A3) by $hv/m$, (A4) by $K$ and adding gives the equation for the time change of kinetic energy

$$\frac{\partial}{\partial t}\left(\frac{h}{mn}K\right) + \frac{\partial}{\partial \xi}\left(\frac{hu}{n}K\right) + \frac{\partial}{\partial \eta}\left(\frac{hv}{m}K\right) + \frac{hu}{n}\frac{\partial \Phi}{\partial \xi} + \frac{hv}{m}\frac{\partial \Phi}{\partial \xi} = 0. \quad (A6)$$

### 2. Derivation of the finite-difference scheme for interior points of the spherical grid

Despite the presence of the metric factors, the way of derivation of a difference scheme for the continuity equation and for the advection and Coriolis terms in the $u$ and $v$ momentum equations closely parallels the methods presented in Section 3. Many of the results of that section will be utilized directly, without repetition of the derivation.

A portion of the spherical grid with the variables staggered as in the C grid and the indices $(i,j)$ centered at a $q$ point is shown in Fig. A1. Here $\Delta\xi$ and $\Delta\eta$ are constant grid intervals in $\xi$ and $\eta$, respectively, and $m$ and $n$ are assumed to vary only in $j$.

For the continuity equation (A4) multiplied by $\Delta\xi\Delta\eta$, the following finite difference form is chosen:

$$\frac{\partial}{\partial t}H_{i+1/2,j+1/2} + u^*_{i+1,j+1/2} - u^*_{i,j+1/2} + v^*_{i+1/2,j+1} - v^*_{i+1/2,j} = 0, \quad (A7)$$

with

$$H_{i+1/2,j+1/2} \equiv \frac{\Delta\xi\Delta\eta}{(mn)_{j+1/2}}h_{i+1/2,j+1/2}, \quad (A8)$$

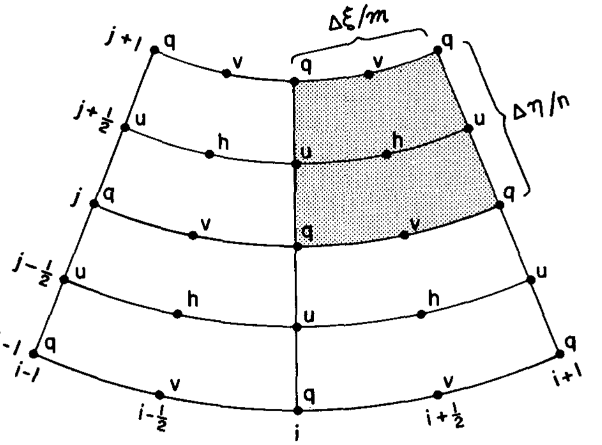


FIG. A1. A portion of the spherical grid used in the derivation of the spherical grid version of the potential enstrophy and energy conserving scheme. The area of the dotted region is represented by $(\Delta\xi\Delta\eta/mn)_{i+1/2,j+1/2}$ (see the text).

$$\left.\begin{aligned} u^*_{i,j+1/2} &\equiv (h^{(u)}u)_{i,j+1/2}\frac{\Delta\eta}{n_{j+1/2}} \\ v^*_{i+1/2,j} &\equiv (h^{(v)}v)_{i+1/2,j}\frac{\Delta\xi}{m_j} \end{aligned}\right\} . \qquad \text{(A9)}$$

Here $\Delta\xi\Delta\eta/(mn)_{j+1/2}$ is area of the stippled region in Fig. A1 and $h^{(u)}$, $h^{(v)}$ are as-yet-unspecified functions of $h$.

Ignoring pressure gradient forces, the remaining terms to be considered can be represented as in the square grid case,

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\Delta\xi}{m_{j+1/2}}u_{i,j+1/2} &- \alpha_{i,j+1/2}v^*_{i+1/2,j+1} - \beta_{i,j+1/2}v^*_{i-1/2,j+1} \\ &- \gamma_{i,j+1/2}v^*_{i-1/2,j} - \delta_{i,j+1/2}v^*_{i+1/2,j} \\ &+ \epsilon_{i+1/2,j+1/2}u^*_{i+1,j+1/2} - \epsilon_{i-1/2,j+1/2}u^*_{i-1,j+1/2} \\ &+ [K_{i+1/2,j+1/2} - K_{i-1/2,j+1/2}] = 0, \end{aligned} \qquad \text{(A10)}$$

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\Delta\eta}{n_j}v_{i+1/2,j} &+ \gamma_{i+1,j+1/2}u^*_{i+1,j+1/2} + \delta_{i,j+1/2}u^*_{i,j+1/2} \\ &+ \alpha_{i,j-1/2}u^*_{i,j-1/2} + \beta_{i+1,j-1/2}u^*_{i+1,j-1/2} \\ &+ \phi_{i+1/2,j+1/2}v^*_{i+1/2,j+1} + \phi_{i+1/2,j-1/2}v^*_{i+1/2,j-1} \\ &+ [K_{i+1/2,j+1/2} - K_{i+1/2,j-1/2}] = 0, \end{aligned} \qquad \text{(A11)}$$

but $u^*$ and $v^*$ are now defined by (A9). The coefficients $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$ and $\phi$ are again functions of $q$ to be determined, where $q_{i,j} \equiv (f+\zeta)_{i,j}/h^{(q)}_{i,j}$ and $h^{(q)}$ is an as-yet-unspecified function of $h$. The finite-difference form for $\zeta$ chosen is

$$\zeta_{i,j} = \frac{(mn)_j}{\Delta\xi\Delta\eta}\left[\left(v\frac{\Delta\eta}{n}\right)_{i+1/2,j} - \left(v\frac{\Delta\eta}{n}\right)_{i-1/2,j} + \left(u\frac{\Delta\xi}{m}\right)_{i,j-1/2} - \left(u\frac{\Delta\xi}{m}\right)_{i,j+1/2}\right]. \qquad \text{(A12)}$$

As in the square grid case, $h^{(u)}$ and $h^{(v)}$ are chosen as

$$\left.\begin{aligned} h^{(u)} &\equiv \bar{h}^i \\ h^{(v)} &\equiv \bar{h}^j \end{aligned}\right\} . \qquad \text{(A13)}$$

Then from the finite-difference analog of (A6) it is easily shown that to maintain conservation of total kinetic energy for divergent mass flux we must specify

$$K_{i+1/2,j+1/2} \equiv \frac{(mn)_{j+1/2}}{\Delta\xi\Delta\eta}\left[\overline{\frac{1}{2}\frac{\Delta\xi\Delta\eta}{m_{j+1/2}n_{j+1/2}}u^2}^{i} + \overline{\frac{1}{2}\frac{\Delta\xi\Delta\eta}{m_jn_j}v^2}^{j}\right]_{i+1/2,j+1/2} . \qquad \text{(A14)}$$

Just as in the square grid case, $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$ and $\phi$ will be determined by constraints on the potential vorticity advection and potential enstrophy conservation for the general case of divergent mass flux. These requirements are imposed through the same procedure as before. The functions $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$ and $\phi$ are expressed in general form as functions of the surrounding $q$ values as in (3.13) but where the coefficients must now be allowed to vary with $\eta$, as do the metric factors. The function $h^{(q)}$ is also defined more generally to allow variation of the weighting factors with $\eta$

$$h^{(q)}_{i,j} = p^1_j(h_{i+1/2,j+1/2} + h_{i-1/2,j+1/2}) + p^2_j(h_{i+1/2,j-1/2} + h_{i-1/2,j-1/2}), \qquad \text{(A15)}$$

where, for consistency, it is required that

$$p^1_j + p^2_j = ½. \qquad \text{(A16)}$$

In addition, when the metric factors are constant in $\eta$, the square grid result $p^1 = p^2 = ¼$ must be recovered.

Applying the constraint on advection of potential vorticity just as in Section 3 gives the requirements

$$A_{j+1/2} = B_{j+1/2} = \frac{(mn)_{j+1/2}}{(mn)_j}p^1_j, \qquad \text{(A18a)}$$

$$C_{j-1/2} = D_{j-1/2} = \frac{(mn)_{j-1/2}}{(mn)_j}p^2_j, \qquad \text{(A18b)}$$

$$E_{j+1/2} = C_{j+1/2} - A_{j+1/2}, \qquad \text{(A18c)}$$

$$F_{j+1/2} = 0 \qquad \text{(A18d)}$$

$$\frac{(mn)_{j-1/2}}{(mn)_{j-1}}p^1_j - \frac{(mn)_{j+1/2}}{(mn)_{j+1}}p^2_{j+1} = \frac{(mn)_{j+1/2}}{(mn)_j}p^1_j - \frac{(mn)_{j-1/2}}{(mn)_j}p^2_j. \qquad \text{(A19)}$$

From (A16) we can express $p^1_j$ and $p^2_j$

$$\left.\begin{aligned} p^1_j &\equiv ¼ + P_j \\ p^2_j &\equiv ¼ - P_j \end{aligned}\right\} , \qquad \text{(A20)}$$

and use (A18a,b,c) and (A19) to solve for the single variable $P_j$:

$$P_j = \frac{1}{(mn)_{j-1/2} + (mn)_{j+1/2}} [\tfrac{1}{4}((mn)_{j-1/2} - (mn)_{j+1/2}) - \tfrac{1}{2}(mn)_j(E_{j-1/2} - E_{j+1/2})]. \quad \text{(A21)}$$

It did not prove necessary to use the variable $E$ in order to achieve a scheme with the desired properties. Thus, as $F$ must vanish, $E$ was chosen to vanish as well for simplicity, and we have from (A21) and (A20)

$$\left.\begin{aligned} p_j^1 &= \frac{1}{2}\frac{(mn)_{j-1/2}}{(mn)_{j-1/2} + (mn)_{j+1/2}} \\ p_j &= \frac{1}{2}\frac{(mn)_{j+1/2}}{(mn)_{j-1/2} + (mn)_{j+1/2}} \end{aligned}\right\} . \quad \text{(A22)}$$

Then

$$\left.\begin{aligned} A_{j+1/2} &= B_{j+1/2} \\ &= \frac{1}{2}\frac{(mn)_{j-1/2}(mn)_{j+1/2}}{(mn)_j[(mn)_{j-1/2} + (mn)_{j+1/2}]} \\ C_{j+1/2} &= D_{j+1/2} \\ &= \frac{1}{2}\frac{(mn)_{j+1/2}(mn)_{j+3/2}}{(mn)_{j+1}[(mn)_{j+1/2} + (mn)_{j+3/2}]} \end{aligned}\right\} , \quad \text{(A23)}$$

so that $A_{j+1/2} = C_{j-1/2}$. But from (A18c) and $E_{j-1/2} = 0$, $A_{j-1/2} = C_{j-1/2}$ and thus $A$ must be independent of $j$. From (A23), then, we must require

$$\frac{(mn)_{j-1/2}(mn)_{j+1/2}}{(mn)_j[(mn)_{j-1/2} + (mn)_{j+1/2}]} = \frac{1}{2}$$

or

$$\frac{\Delta\xi\Delta\eta}{(mn)_j} = \frac{1}{2}\left[\frac{\Delta\xi\Delta\eta}{(mn)_{j-1/2}} + \frac{\Delta\xi\Delta\eta}{(mn)_{j+1/2}}\right] . \quad \text{(A24)}$$

The area $\Delta\xi\Delta\eta/(nm)_j$ for $q$ points must thus be a mean of the areas $\Delta\xi\Delta\eta/(mn)_{j-1/2}$ and $\Delta\xi\Delta\eta/(mn)_{j+1/2}$ for $h$ points.

The constraints imposed up to this point give $A = B = C = D = \tfrac{1}{4}$, $E = F = 0$ and

$$\left.\begin{aligned} p_j^1 &= \frac{1}{4}\frac{(mn)_j}{(mn)_{j+1/2}} \\ p_j^2 &= \frac{1}{4}\frac{(mn)_j}{(mn)_{j-1/2}} \end{aligned}\right\} . \quad \text{(A25)}$$

The procedure for requiring conservation of potential enstrophy is identical to that presented for the square grid and we can use the result given as (3.34) directly, where $q$ is now defined, using (A15) and (A25), as

$$q_{i,j} = \frac{f_j + \zeta_{i,j}}{\dfrac{(mn)_j}{4}\left[\dfrac{h_{i+1/2,j+1/2} + h_{i-1/2,j+1/2}}{(mn)_{j+1/2}} + \dfrac{h_{i+1/2,j-1/2} + h_{i-1/2,j-1/2}}{(mn)_{j-1/2}}\right]}$$

$$= \frac{\dfrac{\Delta\xi\Delta\eta}{(mn)_j}(f_j + \zeta_{i,j})}{\tfrac{1}{4}[H_{i+1/2,j+1/2} + H_{i-1/2,j+1/2} + H_{i+1/2,j-1/2} + H_{i-1/2,j-1/2}]} \quad \text{(A26)}$$

and $\Delta\xi\Delta\eta/(mn)_j$ must satisfy (A24).

### 3. Modification of the scheme at the poles

The poles are singular points of the spherical coordinates and the velocity components cannot be defined there. The poles will thus be specified as $q$ points, for then the velocity at the pole appears in the continuity and momentum equations only through the quotient $v\Delta\xi/m$, which vanishes at the pole, and the modification of the difference equations derived in the previous section is quite straightforward. The staggering of the variables and indexing notation is as shown in Fig. A2, chosen arbitrarily as the North Pole.

The continuity equation at $j = p - \tfrac{1}{2}$ takes the form

$$\frac{\partial H_{i+1/2,p-1/2}}{\partial t} + u^*_{i+1/2,p-1/2} - u^*_{i-1/2,p-1/2} - v^*_{i+1/2,p-1} = 0, \quad \text{(A27)}$$

where

$$H_{i+1/2,p-1/2} = \frac{\Delta\xi\Delta\eta}{(mn)_{p-1/2}} h_{i+1/2,p-1/2}.$$

The representation of the advection terms in the $u$ momentum equation at latitude $p - \tfrac{1}{2}$ and in the $v$ momentum equation at latitude $p - 1$ will be modified by the vanishing of $v^*$ at the pole. However, the following general forms can still maintain the correct cancellation of the terms between equations required in the derivation of the kinetic energy equation

$$\frac{\partial}{\partial t}\left(\frac{u\Delta\xi}{m}\right)_{i,p-1/2} - \gamma'_{i,p-1/2}v^*_{i-1/2,p-1} - \delta'_{i,p-1/2}v^*_{i+1/2,p-1} + \epsilon'_{i+1/2,p-1/2}u^*_{i+1,p-1/2} - \epsilon'_{i-1/2,p-1/2}u^*_{i-1,p-1/2} + [K'_{i+1/2,p-1/2} - K'_{i-1/2,p-1/2}] = 0, \quad \text{(A28)}$$

$$\frac{\partial}{\partial t}\left(\frac{v\Delta\eta}{n}\right)_{i+1/2,p-1} + (\gamma' u^*)_{i+1,p-1/2} + (\delta' u^*)_{i,p-1/2} + (\alpha u^*)_{i,p-3/2} + (\beta u^*)_{i+1,p-3/2} - \phi_{i+1/2,p-2} v^*_{i+1/2,p-2} + [K'_{i+1/2,p-1/2} - K_{i+1/2,p-3/2}] = 0. \quad \text{(A29)}$$

In the above equations, the primed variables are to be determined; the others are defined as in (3.34).

At the poles, we define $q$ through the circulation theorem

$$q_p \equiv \frac{f_p + \zeta_p}{h_p^{(q)}}, \quad \text{(A30)}$$

where

$$\zeta_p = \frac{1}{A_p^q} \sum_{i=1}^{\text{IMAX}} \left(\frac{u\Delta\xi}{m}\right)_{i,p-1/2}, \quad \text{(A31)}$$

$$A_p^q = \text{IMAX}\frac{\Delta\xi\Delta\eta}{(mn)_p}. \quad \text{(A32)}$$

The factor $\Delta\xi\Delta\eta/(mn)_p$ in (A32) represents the area of the hatched region in Fig. A2, and IMAX is the total number of grid points in the $\xi$ direction.

The formation of the kinetic energy equation at $j = p - ½$ leads to the result that for kinetic energy conservation in the divergent mass flux case we must define

$$K'_{i+1/2,p-1/2} \equiv \frac{1}{2}\frac{mn_{p-1/2}}{\Delta\xi\Delta\eta}\left[\overline{\frac{\Delta\xi}{m_{p-1/2}}\frac{\Delta\eta}{n_{p-1/2}}u^2_{i,p-1/2}}^{i} + \frac{1}{2}\frac{\Delta\xi}{m_{p-1}}\frac{\Delta\eta}{n_{p-1}}v^2_{i+1/2,p-1}\right]. \quad \text{(A33)}$$

It remains to apply the constraints on potential vorticity advection and potential enstrophy conservation at $j = p - 1$ and $j = p$. For advection of potential vorticity, as before, we specify $h^{(q)}$ and require that the potential vorticity equation be consistent with the continuity equation when $q$ is constant in space. At $j = p - 1$,

$$h^{(q)}_{i,p-1} = p_p^{(1)}(h_{i+1/2,p-1/2} + h_{i-1/2,p-1/2}) + p_p^{(2)}(h_{i+1/2,p-3/2} + h_{i-1/2,p-3/2}), \quad \text{(A34)}$$

and after making use of the values (3.34) for interior points and taking $E' = 0$, we obtain

$$p_p^{(1)} = \frac{1}{4}\frac{(mn)_{p-1}}{(mn)_{p-1/2}}, \quad p^{(2)} = \frac{1}{4}\frac{(mn)_{p-1}}{(mn)_{p-3/2}}, \quad \text{(A35)}$$

and $C' = D' = ¼$. The requirement for consistency in (A34) gives the constraint on the area-weighting factors

$$\frac{\Delta\xi\Delta\eta}{(mn)_{p-1}} = \frac{1}{2}\left[\frac{\Delta\xi\Delta\eta}{(mn)_{p-1/2}} + \frac{\Delta\xi\Delta\eta}{(mn)_{p-3/2}}\right]. \quad \text{(A36)}$$

To apply the same constraint on the advection

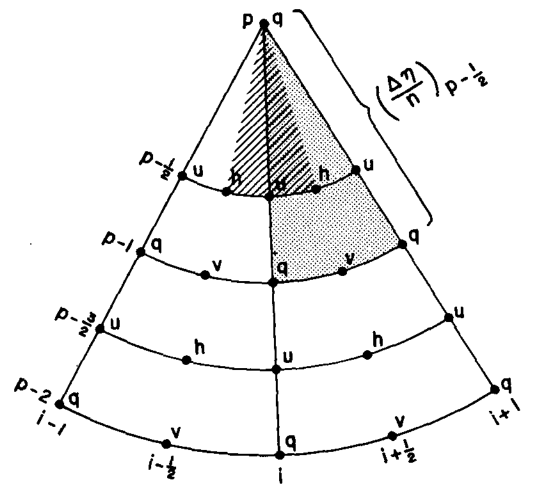


FIG. A2. The spherical grid near the North Pole used in the derivation of the spherical grid version of the potential enstrophy and energy conserving scheme. The areas of the dotted and hatched regions respectively represent $(\Delta\xi\Delta\eta/mn)_p$ and $(\Delta\xi\Delta\eta/mn)_{p-1/2}$ (see the text).

of potential vorticity at the pole, we let $h^{(q)}$ be a function of all IMAX $h$ values at $p - ½$, such that

$$h_p^{(q)} \equiv \sum_{i=1}^{\text{IMAX}} p_p^{(1)} h_{i+1/2,p-1/2}, \quad \text{(A37)}$$

where

$$\sum_{i=1}^{\text{IMAX}} p_p^{(1)} = 1. \quad \text{(A38)}$$

When we require consistency between the vorticity and continuity equations at the pole, it turns out that

$$p_p^{(1)} = \frac{1}{\text{IMAX}}, \quad \text{(A39)}$$

$$\frac{\Delta\xi\Delta\eta}{(mn)_p} = \frac{1}{2}\frac{\Delta\xi\Delta\eta}{(mn)_{p-1/2}}. \quad \text{(A40)}$$

The area factors $\Delta\xi\Delta\eta/(mn)_p$ and $\Delta\xi\Delta\eta/(mn)_{p-1/2}$ in (A40), on the other hand, must approximate the geometrical areas of the hatched and stippled regions in Fig. A2, respectively. It is then desirable to choose a grid such that (A40) is approximately satisfied by the geometrical areas. Thus the distance $(\Delta\eta/n)_{p-1/2}$ in Fig. A2 is chosen to be 3/2 of $\Delta\eta/n$ for the interior part of the grid.

Turning now to the requirement of potential enstrophy conservation, we must form the equations for time change of potential enstrophy at $j = p - 1$ and $j = p$ and require that they interact with the similar equation at $j = p - 2$ and with each other to give a vanishing global sum. A lengthy manipulation following the now familiar procedure finally gives

$$\left.\begin{aligned} \gamma'_{i,p-1} &= \tfrac{1}{24}[3q_p + 2q_{i-1,p-3/2} + q_{i,p-3/2}] \\ \delta'_{i,p-1} &= \tfrac{1}{24}[3q_p + q_{i,p-3/2} + 2q_{i+1,p-3/2}] \\ \epsilon'_{i+1/2,p-1} &= \tfrac{1}{24}[2q_p - q_{i,p-3/2} - q_{i+1,p-3/2}] \end{aligned}\right\} . \quad \text{(A41)}$$

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