Exponentially Unstable Edge Modes at Large Froude Number: the Inertial-Gravity-Edge Instability

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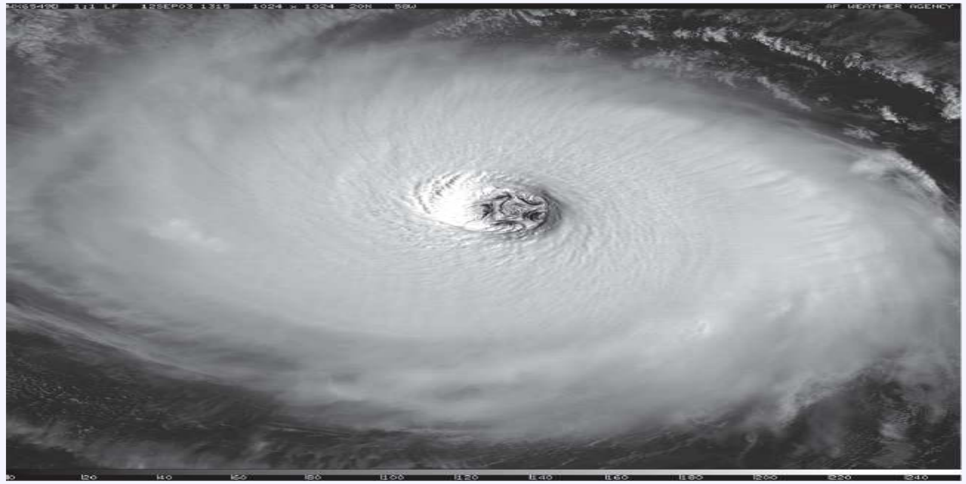
Overview

Hurricane Isabel, 12 Sept. 2003



- Images of polygonal hurricane eyewalls have spurred interest in wave-vortex coupling—role in intensification?
- Established models of linear instability (e.g. Montgomery, Schubert, Nolan, Schecter etc.):
  - Centrifugal (symmetric) instability: fast; observed?
  - Vortex Rossby Waves (VRW): barotropic VRW-VRW coupling.
  - Rossby-Inertial-Buoyancy (RIB): baroclinic VRW-IB coupling.

Overview (II)

Properties of the RIB instability:

- Observed to require a sharp negative PV gradient, arguably where the VR and IB waves couple.
- First studied at small Froude number by Ford (1994) who concluded:
  “unlikely to be of any practical interest in geophysical fluid dynamics”
- Relevant in the “superspin” regime of large Rossby ($\mathrm{R_0}$) and Froude ($\mathrm{F_r}$) numbers, e.g. hurricanes.
- Attractive $\longrightarrow$ unstable at wave-number 2 which is “missing” in VRW-VRW models (Terwey & Montgomery, 2002).

However, Rossby character of RIB instability has never been formally established at large $\mathrm{F_r}$.

Outline

Investigate the nature of the 3D instability that lives on a piece-wise constant vorticity profile (Schubert profile) using three models:

Model # 1:

- Construct analytic normal mode solutions.
- Solve for the eigenvalue (frequency) numerically.

Model # 2:

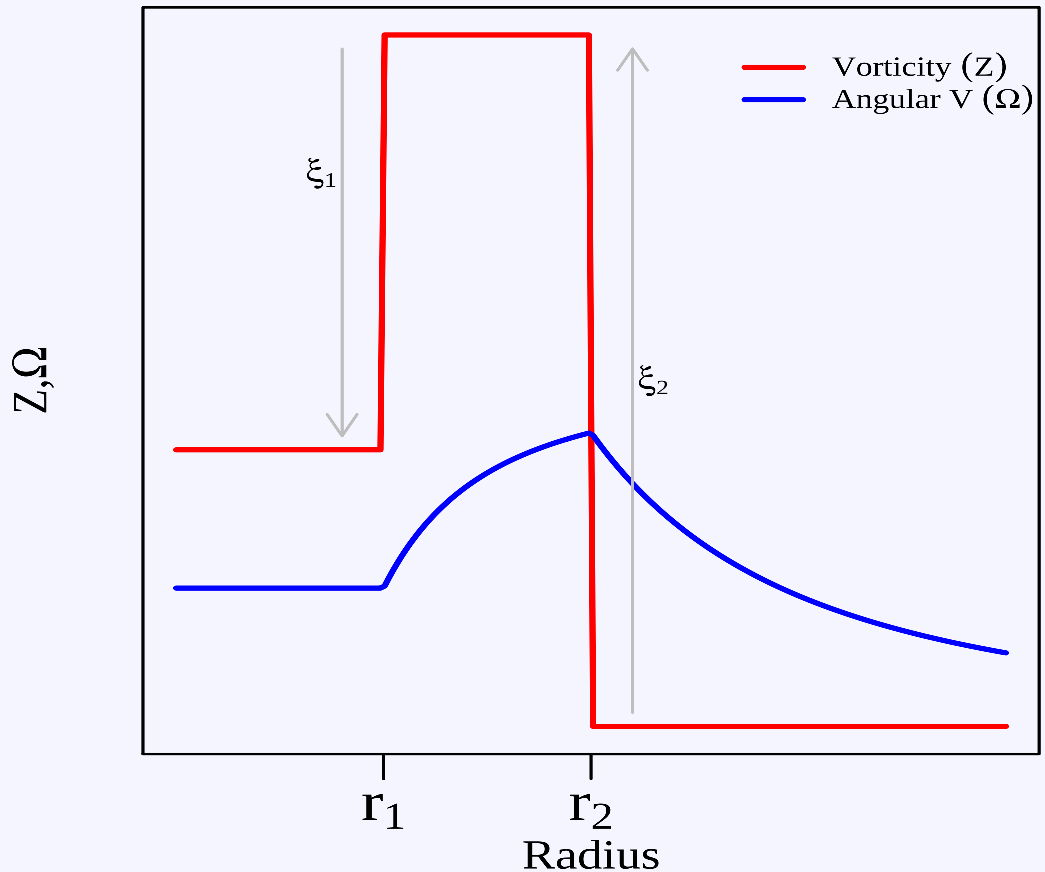
- Derive an approximate analytic dispersion relation for baroclinic modes on the Schubert base-state.

Model # 3:

- Solve the initial value problem for the linearized primitive equations numerically.

Base State

Piecewise constant vorticity (Schubert 1999):



Normal Mode Solutions

$$\phi(r,\theta,z,t) = \hat{\phi}(r)\exp\left[i(l\theta + mz - \nu t)\right]$$

Parameters:

$$\begin{aligned} l &= \text{Azimuthal wave-number} \\ m &= \text{Vertical wave-vector} \\ \nu &= \text{Frequency} \\ \{r_1, r_2, \xi_1, \xi_2\} &= \text{Base-state Vorticity} \\ \Omega(r) &= \text{Base-state Angular Velocity} \\ N &= \text{Brunt-Vaisala Frequency} \end{aligned}$$

Non-Dimensional Numbers:

$$\mathrm{R}_0 = \frac{\xi_2}{f} = \text{Rossby Number}$$

$$\mathrm{F}_\mathrm{r} = \frac{r_2 m \xi_2}{N} = \text{Froude Number}$$

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Model #1: 3-Region Analytic Model Numerical Eigenvalues

Analytic Model

Pressure equation from linearized primitive equations [e.g. Smyth and McWilliams, 1998]:

$$\left(\frac{r\pi'}{\Phi_a-\sigma^2}\right)' + \left[-\frac{l}{\sigma}\left(\frac{2\Omega+f}{\Phi_a-\sigma^2}\right)' - \frac{l^2}{r(\Phi_a-\sigma^2)} - \frac{m^2 r}{N^2}\right]\pi = 0$$

where

$$\begin{aligned} \sigma &= \nu - l\Omega(r) \\ \Phi_a &= (2\Omega(r)+f)(Z+f) \\ &= \text{Absolute centrifugal stability} \end{aligned}$$

Step 1: Define

$$G = \frac{\pi' r}{\Phi_a - \sigma^2}$$

and take limit $\mathrm{F_r} \rightarrow \infty$.

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Analytic Model (II)

Equation set:

$$G = \frac{\pi' r}{\Phi_a - \sigma^2}$$

$$0 = r^3 \left(\frac{G'}{r}\right)' - \frac{m^2 r^2}{N^2}(\Phi_a - \sigma^2) G$$

with polarization relations:

$$u_r = i\frac{\sigma}{r}G - \frac{il}{r^2}\frac{(2\Omega + f)}{(\Phi_a - \sigma^2)}\frac{N^2}{m^2}G'$$

$$u_\theta = (Z + f)\frac{G}{r} - \frac{l}{r^2}\frac{\sigma}{(\Phi_a - \sigma^2)}\frac{N^2}{m^2}G'$$

$$u_z = -\frac{\sigma G'}{mr}$$

Note: exact momentum balance, approximate incompressibility.

Solution

Region I ($r \le r_1$):

$$\pi = J_l\left(\sqrt{\frac{\sigma^2 - \Phi_a}{N^2}}mr\right)$$

Region II ($r_1 < r \le r_2$):

$$\begin{aligned} r^2(\sigma^2 - \Phi_a)^2 &= a_1 r^2 + a_2 + a_3 r^{-2} \\ &\approx a_1 r^2 + a_2 + a_3[(r_1 + r_2)/2]^{-2} \end{aligned}$$

Resulting equation is

$$r^2 G'' - rG' + (C + Br^2)G = 0$$

with solution

$$G = r\left[c_1 J_\nu(\sqrt{B}r) + c_2 Y_\nu(\sqrt{B}r)\right]$$

and $\nu = \sqrt{1 - C}$.

Solution (II)

Region III ($r > r_2$):

$$r^2(\sigma^2 - \Phi_a)^2 = a_1 r^2 + a_2 + a_3 r^{-2}$$
$$\approx a_1 r^2 + a_2 + a_3 \left(2r_2^{-2} - r^2 r_2^{-4}\right)$$

Resulting equation is

$$r^2 G'' - rG' + (C + Br^2)G = 0$$

with solution (radiation boundary condition)

$$G = rH_\nu(\sqrt{B}r)$$

and $\nu = \sqrt{1 - C}$.

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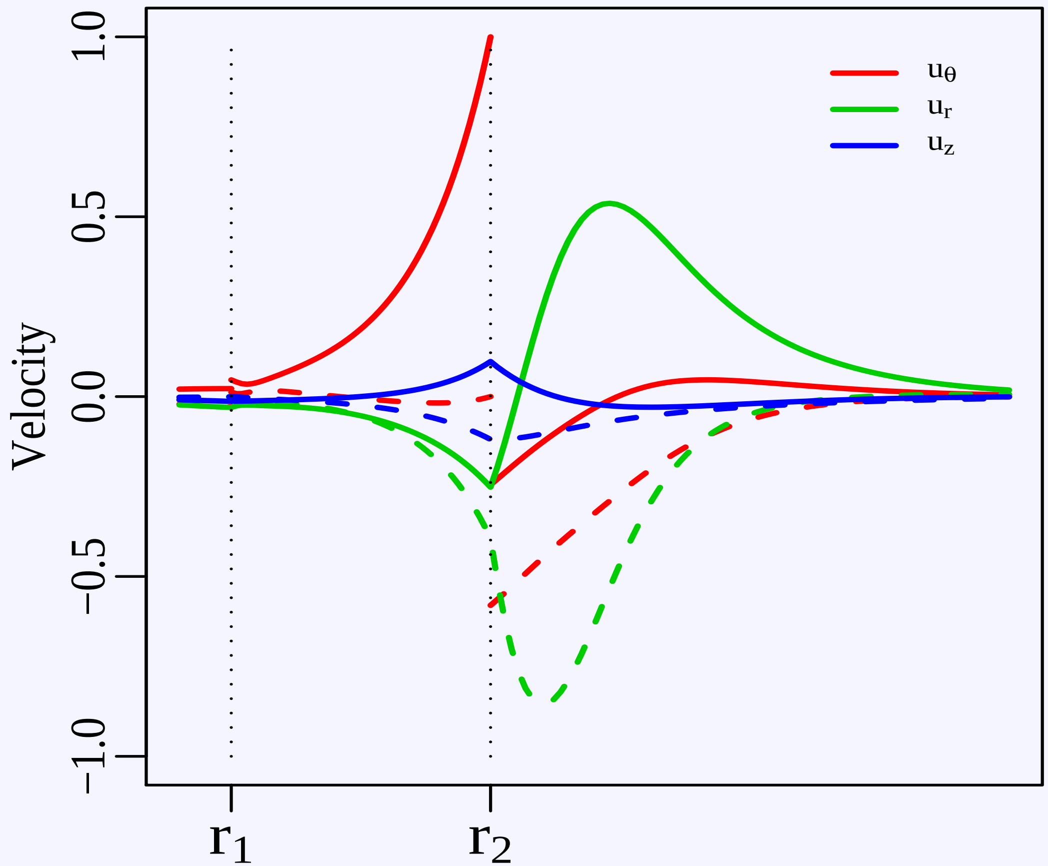
Example: Exponentially Unstable Mode

Solve eigenvalue problem for jump conditions at (r_1, r_2).

$$\lim_{\epsilon \to 0} u_r(r_1 - \epsilon) = u_r(r_1 + \epsilon), \pi(r_1 - \epsilon) = \pi(r_1 + \epsilon)$$



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Features of Unstable Modes

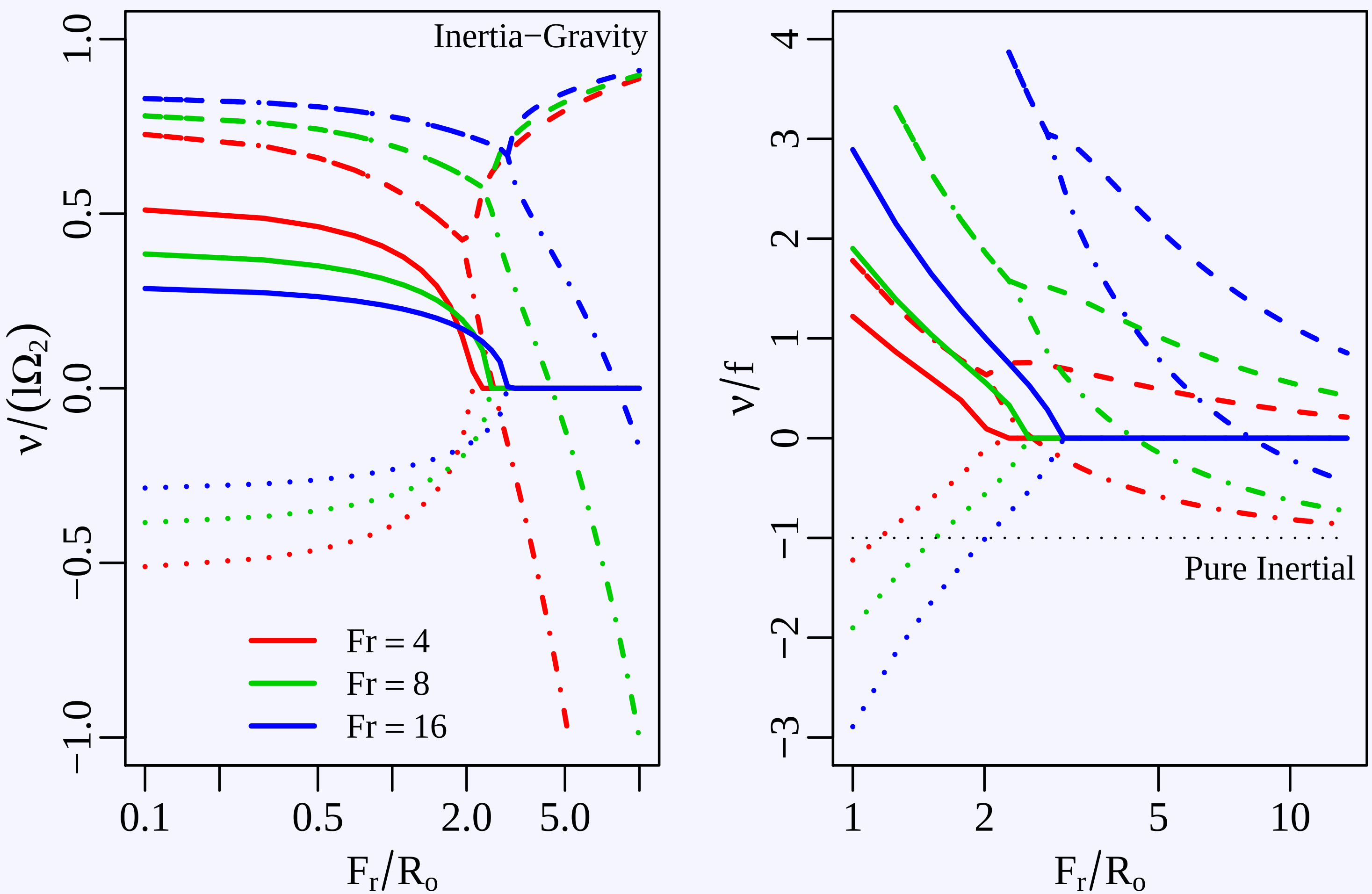
- Instability:
  Modes become unstable when (equivalently)

$$r_2 \ll \text{Rossby Deformation Radius}$$
$$\text{Rotational Froude Number} \ll 1$$

- Frequency:

$$\begin{aligned} \mathrm{Im}(\nu) &\sim \mathrm{F_r}^{-1/2} \\ \mathrm{Re}(\nu) - l\Omega(r_2) &\sim \mathrm{F_r}^{-1/2} \end{aligned}$$

Phase Diagram



Unstable wave couples a retrograde inertial oscillation and a prograde gravity wave.

Model #2: Approximate Analytic Dispersion Relation

Dispersion Relation

Analysis of the jump conditions at r_2 suggests an edge wave BC:

$$u_r(r_2)\Big|^{r_2+\epsilon} = 0$$

Using the WKB approximation to relate π' to π we find the dispersion relation:

$$\nu = l\Omega_2 - \frac{1}{\sqrt{2}}\frac{\sqrt{N(2\Omega_2+f)}}{\sqrt{r_2 m}}\sqrt{\frac{r_2}{\mathrm{L_d}} \pm \sqrt{\left(\frac{r_2}{\mathrm{L_d}}\right)^2 - 4l^2}}$$

with stability criterion

$$r_2 < 2l\mathrm{L_d}$$

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Dispersion Relation (II)

Unstable: ($r < 2l\mathrm{L_d}$)

$$\mathrm{Im}(\nu) = \pm\frac{1}{\sqrt{2}}\frac{\sqrt{lN(2\Omega_2 + f)}}{\sqrt{r_2 m}} \sim \frac{1}{\sqrt{\mathrm{F_r}}}$$

Stable: ($r > 2l\mathrm{L_d}$)

Wave 1:

$$\nu = l\Omega_2 - \sqrt{2\Omega_2 + f}\sqrt{f}$$

This is an inertial oscillation that is retrograde at small Rossby number.

Wave 2:

$$\nu = l\Omega_2 - \sqrt{\frac{2\Omega_2 + f}{f}\frac{lN}{r_2 m}}$$

This is an inertia-gravity wave that is prograde at large Froude number.

These waves phase-lock and grow when $r_2 = l\mathrm{L_d}$

Interpretation

Very large body of literature on the non-centrifugal (non-symmetric) instability of stably stratified vortices with sharp vorticity gradients (Kurihara, Willoughby, Ford).

This work: **Inertia-Gravity-Edge Instability**

- Outer vorticity gradient acts as a moving edge ($u_r = 0$). The total centrifugal force $\sim 2\Omega_2 + f$ at r_2 supports the waves.
- Mode is similar to stratified Taylor-Couette instability (Yavneh, McWilliams, Molemaker), with **no Rossby character**.

Modern Interpretation (Schecter, Montgomery, Hodyss, Nolan etc.): **Rossby-Inertia-Buoyancy Instability**

- RIB instability couples an inner Vortex Rossby wave with an outer inertia-buoyancy wave.
- Requires relatively sharp negative PV gradient for instability.

Test of Two Hypotheses

Present Analysis (large $\mathrm{F_r}$):

$$\mathrm{Im}(\nu) \sim \frac{\sqrt{2\Omega_2 + f}}{\sqrt{\mathrm{F_r}}}, \text{ Independent of } \xi_2$$

Ford (1994; small $\mathrm{F_r}$):

$$\mathrm{Im}(\nu) \sim \mathrm{F_r}^{2l} \left(l\Omega_2 - \frac{\xi_2}{2} \right)^{2l} + \mathcal{O}(\mathrm{F_r}^{2l+2})$$

Experiment: Using the linearized primitive equations, we solve initial-value problem for the Schubert base-state:

Case 1: Vary the Vorticity Jump $\xi_2(r_1, \xi_1)$ keeping the Angular Velocity Ω_2 fixed.

Case 2: Vary the Angular Velocity $\Omega_2(r_1, \xi_1)$ keeping the Vorticity Jump ξ_2 fixed.

Hypothesis: VRW coupling will exhibit ξ_2-dependence.

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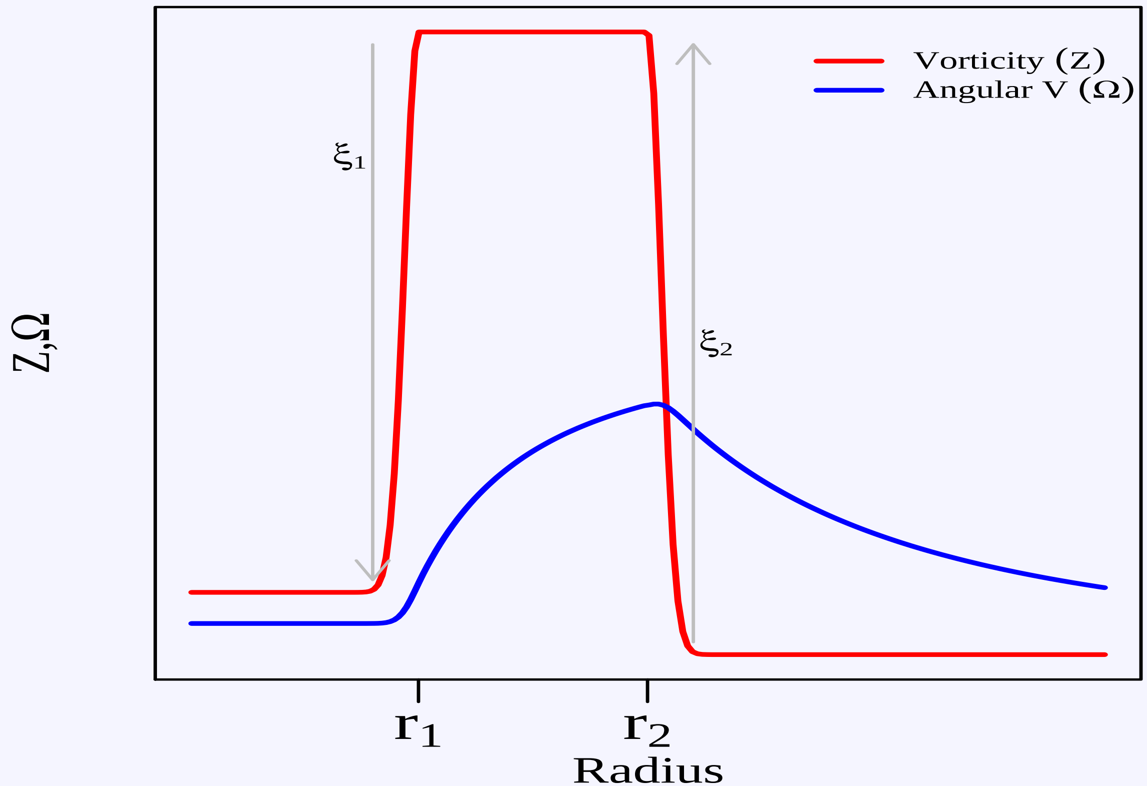
Model #3: Numerical Primitive Equation Model

Simulations & Numerics

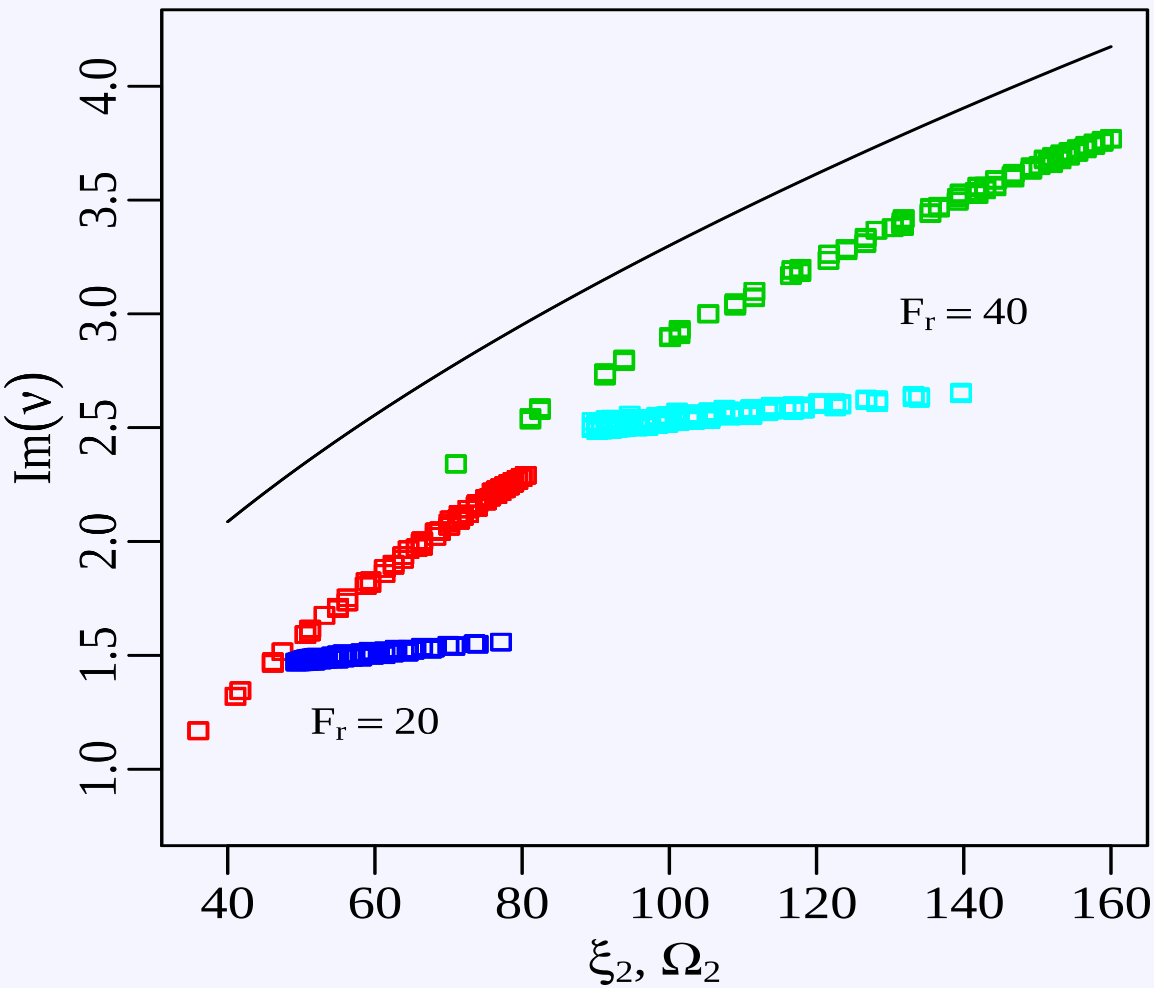
Base-state:



Numerics:

- Strong Stability Preserving 3rd-Order RK (SSP33).
- PV conserving numerical scheme.

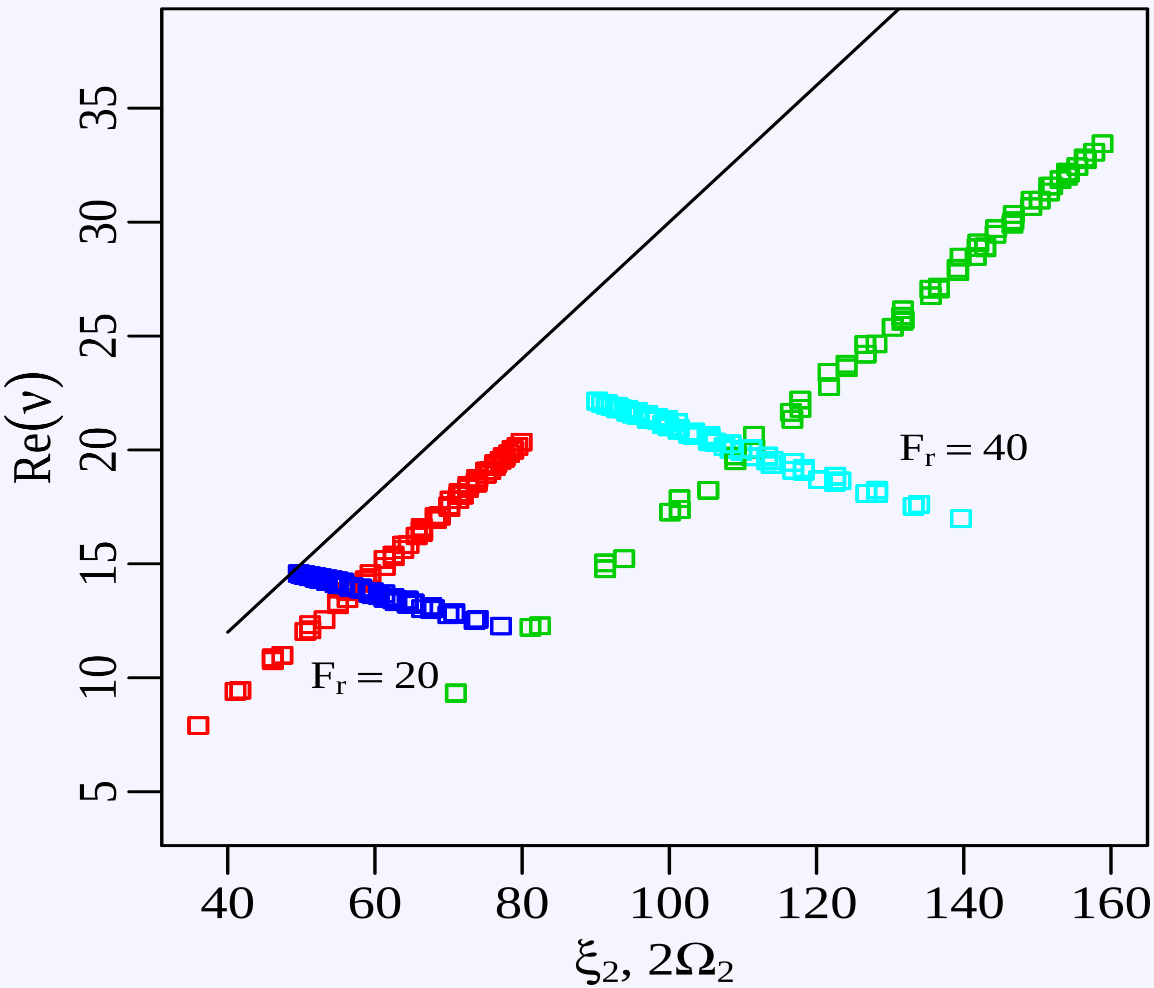
Results



Approximate numerical result:

$$\mathrm{Im}(\nu) \sim (2\Omega + 1)^{0.6} \xi_2^{0.1}$$

Results (II)



Approximate numerical result:

$$\mathrm{Re}(\nu) \sim (2\Omega + 1)^{1.2}\xi_2^{-0.5}$$

Summary

- Investigated unstable modes that grow on vorticity gradients.
- In the superspin regime of large Froude and Rossby numbers:

> Inertial-Gravity-Edge Instability: coupling of a retrograde inertial oscillation and a prograde gravity wave.

- Approximate dispersion relation:

$$\nu = l\Omega_2 - \frac{1}{\sqrt{2}} \frac{\sqrt{N(2\Omega_2 + f)}}{\sqrt{r_2 m}} \sqrt{\frac{r_2}{\mathrm{L_d}} \pm \sqrt{\left(\frac{r_2}{\mathrm{L_d}}\right)^2 - 4l^2}}$$

- We find little or no evidence of a Rossby-wave character (vorticity gradient dependence) in the growth rate.
- However, the oscillation frequency does have a mild vorticity-gradient dependence; this is currently being investigated.

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